Linearisation of Ideal MHD

Contents

- Small perturbations
- Linear equations
- Wave and unstable solutions

$$\frac{\partial \rho}{\partial t} + \underline{u}\cdot\nabla\rho = -\rho\nabla\cdot\underline{u} \qquad \frac{\partial \underline{B}}{\partial t} = \nabla\times(\underline{u}\times\underline{B})$$

$$\rho\left[\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right] = \underline{-\nabla p + \underline{j}\times\underline{B}} \qquad \underline{j} = \frac{1}{\mu_0}\nabla\times\underline{B}$$

$$\frac{\partial p}{\partial t} + \underline{u}\cdot\nabla p = -\gamma p\nabla\cdot\underline{u}$$

$$\rho = \overset{\text{equilibrium}}{\rho_0} + \delta\rho(\underline{x},t) \qquad \delta\rho \ll \rho_0$$

$$\uparrow$$

$$\frac{\partial \rho_0}{\partial t} = 0$$

$$\frac{\partial}{\partial t}(\cancel{\rho_0} + \delta\rho) + \cancel{u_0\cdot\nabla\rho_0} = -\cancel{\rho_0\nabla\cdot\underline{u}_0} \qquad \text{equilibrium}$$

$$\boxed{+\,\delta\underline{u}\cdot\nabla\rho_0 + u_0\cdot\nabla\delta\rho = -\delta\rho\nabla\cdot\underline{u}_0 - \rho_0\nabla\cdot\delta\underline{u} \quad \text{linear}}$$

$$\text{nonlinear}\ \left[+\,\delta u_0\cdot\nabla\delta\rho_0\right] = \left[-\,\delta\rho_0\nabla\cdot\delta u_0\right]$$

small

$$\frac{\partial}{\partial t}\delta\rho + \delta u\cdot\nabla\rho_0 + u_0\cdot\nabla\delta\rho = -\delta\rho\nabla\cdot\underline{u}_0 - \rho_0\nabla\cdot\delta u$$

$$* \quad \frac{\partial \delta \underline{u}}{\partial t} = -\underline{u}_0 \cdot \nabla \delta \underline{u} - \delta \underline{u} \cdot \nabla \underline{u}_0 - \frac{1}{\rho_0} \nabla \delta p + \frac{1}{\mu_0 \rho_0}\left[(\nabla \times \underline{B}_0) \times \delta \underline{B} + (\nabla \times \delta \underline{B}) \times \underline{B}_0 \right]$$

$$\frac{\partial \delta p}{\partial t} = -\delta \underline{u} \cdot \nabla p_0 - u_0 \cdot \nabla \delta p - \gamma \delta p \nabla \cdot \underline{u}_0 - \gamma p_0 \nabla \cdot \delta \underline{u}$$

$$\frac{\partial \delta \underline{B}}{\partial t} = \nabla \times (\delta \underline{u} \times \underline{B}) + \nabla \times (u_0 \times \delta \underline{B})$$

$$\frac{\partial}{\partial t}\begin{pmatrix} \delta \rho_0 \\ \delta \underline{u} \\ \delta p \\ \delta \underline{B} \end{pmatrix} = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}\begin{pmatrix} \delta \rho_0 \\ \delta \underline{u} \\ \delta p \\ \delta \underline{B} \end{pmatrix}$$

$$\frac{\partial f}{\partial t} = \alpha f \quad \Rightarrow \quad f \propto e^{i\omega t}$$

ω frequency

if ω imaginary $\rightarrow$ growing or shrinking

here $\underline{u}_0 = 0$

$$\frac{\partial^2 \delta \underline{u}}{\partial t^2} = -\frac{1}{\rho_0}\underbrace{\frac{\partial \delta p}{\partial t}}_{} + \frac{1}{\mu_0 \rho_0}\left[(\nabla \times \underline{B}_0) \times \underbrace{\frac{\partial \delta \underline{B}}{\partial t}}_{} + \left(\nabla \times \underbrace{\frac{\partial \delta \underline{B}}{\partial t}}_{}\right) \times \underline{B}_0 \right]$$

$$-\omega^2 \delta \underline{u} = -\frac{1}{\rho_0}\left[-\delta \underline{u} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \delta \underline{u} \right]$$

c_s^2 sound

$$+ \frac{1}{\mu_0 \rho_0}\left\{ (\nabla \times \underline{B}_0) \times [\nabla \times (\delta \underline{u} \times \underline{B}_0)] + [\nabla \times \nabla \times (\delta u \times \underline{B}_0)] \times \underline{B}_0 \right]$$

$B_0^2 / \mu_0 \rho_0 \simeq V_A^2$ Alfven speed