Normal modes

Contents

- Derive linear equation for displacements
- Show that frequency is purely real or imaginary
- Show that discrete modes are orthogonal

$$\rho = \rho_0 + \delta\rho \qquad \underset{\sim}{u}_0 = 0 \qquad \underset{\sim}{J}_0 \times \underset{\sim}{B}_0 = \nabla P_0$$

ρ_0: equilibrium; $\delta\rho$: small perturbed quantities; $\underset{\sim}{u}_0 = 0$: stationary

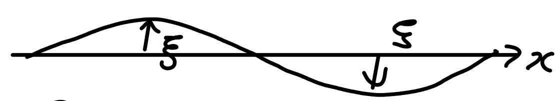
$$\int \frac{\partial \delta\rho}{\partial t}\,dt = \int \Big(\underbrace{-\delta \underset{\sim}{u}\cdot\nabla\rho_0}_{\to\, -\underset{\sim}{\xi}\cdot\nabla\rho_0} \underbrace{-\,\rho_0 \nabla\cdot\delta\underset{\sim}{u}}_{-\rho_0\nabla\cdot\underset{\sim}{\xi}} \Big)\,dt$$

$$\frac{\partial \delta P}{\partial t} = -\delta\underset{\sim}{u}\cdot\nabla P_0 - \gamma P_0 \nabla\cdot\delta\underset{\sim}{u}$$

$$* \quad \frac{\partial \delta \underset{\sim}{u}}{\partial t} = -\frac{1}{\rho_0}\nabla \delta P + \frac{1}{\mu_0\rho_0}\Big[(\nabla\times\underset{\sim}{B}_0)\times\delta\underset{\sim}{B} + (\nabla\times\delta\underset{\sim}{B})\times\underset{\sim}{B}_0 \Big]$$

$$\frac{\partial \delta\underset{\sim}{B}}{\partial t} = \nabla\times(\delta\underset{\sim}{u}\times\underset{\sim}{B})$$

Displacement $\underset{\sim}{\xi}(\underset{\sim}{x}, t)$



$$\delta\underset{\sim}{u} = \frac{\partial}{\partial t}\underset{\sim}{\xi}$$

$$\delta\rho = -\underset{\sim}{\xi}\cdot\nabla\rho_0 - \rho_0\nabla\cdot\underset{\sim}{\xi}$$

$$\delta P = -\underset{\sim}{\xi}\cdot\nabla P_0 - \gamma P_0 \nabla\cdot\underset{\sim}{\xi}$$

$$\delta\underset{\sim}{B} = \nabla\times(\underset{\sim}{\xi}\times\underset{\sim}{B})$$

$$\rho_0 \frac{\partial^2 \underline{\xi}}{\partial t} = -\nabla \delta p + \frac{1}{\mu_0}\left[(\nabla \times \underline{B}_0) \times \delta \underline{B} + (\nabla \times \delta \underline{B}) \times \underline{B}_0 \right]$$

$$= \nabla \left(\underline{\xi} \cdot \nabla P_0 + \gamma P_0 \nabla \cdot \underline{\xi} \right)$$

$$+ \frac{1}{\mu_0}\left[(\nabla \times \underline{B}_0) \times \nabla \times (\underline{\xi} \times \underline{B}) + \nabla \times \nabla \times (\underline{\xi} \times \underline{B}) \times \underline{B}_0 \right]$$

$$= F(\underline{\xi})$$ Force operator

Linear function $\quad \xi = \hat{\xi}(\underline{x}) e^{-i\omega t} \quad$ ω frequency

$$* \quad \underline{\underline{-\rho_0 \omega^2 \hat{\underline{\xi}} = F(\hat{\underline{\xi}})}}$$ Normal mode formulation

<u>Properties of $F(\hat{\underline{\xi}})$</u>

$$\int \hat{\eta}(\underline{x}) F(\hat{\xi}(\underline{x}))\, d\underline{x} = \int \hat{\xi}(x) F(\hat{\eta}(x))\, d\underline{x}$$

① ω^2 real

Multiply $*$ by $\hat{\underline{\xi}}^*$

$$-\rho_0 \omega^2 |\xi|^2 = \xi^* F(\hat{\xi}) \qquad Ⓐ$$

$$-\rho_0 (\omega^2)^* \underline{\xi}^* = F(\xi^*)$$

$$\Rightarrow -\rho_0 (\omega^2)^* |\xi|^2 = \xi F(\xi^*) \qquad Ⓑ$$

(A) $\quad -\omega^2 \int \rho_0 |\underline{\xi}|^2 d\underline{x} = \int \xi^* F(\xi)\, d\underline{x}$

(B) $\quad -(\omega^2)^* \int \rho |\xi|^2 dx = \int \xi\, F(\xi^*)\, d\underline{x}$

equal

Self-adjoint

$\omega^2 = (\omega^2)^*$

$\Rightarrow \omega^2$ is real

waves are either

- Growing if $\omega^2 < 0$
- Oscillating if $\omega^2 < 0$

$\Rightarrow$ Marginal stability can be found where
$\omega^2 = 0$

② Orthogonal modes

$$-\rho_0 \omega_n^2 \underline{\xi} = F(\xi) \qquad -\rho_0 \omega_m^2 \underline{\eta} = F(\eta)$$

$$-\rho_0 \omega_n^2 \underline{\eta} \cdot \underline{\xi} = \eta \cdot F(\xi) \qquad -\rho_0 \omega_m^2 \underline{\xi} \cdot \underline{\eta} = \underline{\xi} \cdot F(\eta)$$

$$-\omega_n^2 \int \rho_0 \underline{\eta} \cdot \underline{\xi}\, d\underline{x} = \underline{\int \eta \cdot F(\xi)\, d\underline{x}} \qquad -\omega_m \int \rho_0\, \underline{\xi} \cdot \eta\, dx = \underline{\int \underline{\xi} \cdot F(\eta)\, dx}$$

$$(\omega_n - \omega_m) \int \rho_0\, \underline{\eta} \cdot \underline{\xi}\, d\underline{x} = 0$$

$\Rightarrow$ if $\omega_n \neq \omega_m \quad \Rightarrow \int \rho_0\, \eta \cdot \underline{\xi}\, dx = 0$

if ρ_0 const $\rightarrow$ orthogonal

Discrete modes are decoupled