# Incompressibility

Contents

- Minimisation of delta-W (variational method)
- Singular surfaces

$\xi_{\parallel} = \underset{\sim}{b} \cdot \underset{\sim}{\xi}$

$$\delta W_p = \frac{1}{2}\int d\underset{\sim}{x} \Big[ \underbrace{\frac{|\delta B_{\perp}|^2}{\mu_0} + \frac{B_0^2}{\mu_0}\left|\nabla\cdot\underline{\xi}_{\perp} + \underline{\xi}_{\perp}\cdot\underset{\sim}{\kappa}\right|^2 + \gamma P_0 |\nabla\cdot\underset{\sim}{\xi}|^2 - 2(\underline{\xi}_{\perp}\cdot\nabla P_0)(\underset{\sim}{\kappa}\cdot\underline{\xi}_{\perp}^{*}) - \delta\underset{\sim}{B}\cdot(\underline{\xi}_{\perp}\times\underset{\sim}{b})J_{\parallel}}_{\mathcal{L}(\underset{\sim}{x})} \Big]$$

Variational method

Minimise $\delta W_p$ w.r.t. variations in $\xi_{\parallel}$

Euler - Lagrange
$$\frac{\partial \mathcal{L}}{\partial \xi_{\parallel}} - \nabla\cdot\frac{\partial \mathcal{L}}{\partial \nabla \xi_{\parallel}} = 0$$

$$\mathcal{L} = \gamma P_0 |\nabla\cdot\underset{\sim}{\xi}|^2 = \gamma P_0 (\nabla\cdot\underset{\sim}{\xi})(\nabla\cdot\xi^{*})$$

$$\underset{\sim}{\xi} = \underline{\xi}_{\perp} + \underset{\sim}{b}\,\xi_{\parallel}$$

$$\mathcal{L} = \gamma P_0 \left(\nabla\cdot\underline{\xi}_{\perp} + \nabla\cdot(b\xi_{\parallel})\right)(\nabla\cdot\underline{\xi}^{*})$$

$$= \gamma P_0 \left(\nabla\cdot\xi_{\perp} + \underline{\underset{\sim}{b}\cdot\nabla\xi_{\parallel}} + \underline{\xi_{\parallel}\nabla\cdot b}\right)(\nabla\cdot\xi^{*}) \longleftarrow$$

$$\frac{\partial \mathcal{L}}{\partial \xi_{\parallel}} = \gamma P_0 (\nabla\cdot\underset{\sim}{b})(\nabla\cdot\xi^{*}) + \underline{\gamma P_0 (\nabla\cdot\underset{\sim}{b})(\nabla\cdot\underline{\xi})}$$

$$\frac{\partial \mathcal{L}}{\partial \nabla \xi_{\parallel}} = \gamma P_0 (\nabla\cdot\xi^{*})\underset{\sim}{b} + \underline{\gamma P_0 (\nabla\cdot\underline{\xi})\underset{\sim}{b}}$$

$$\frac{\partial L}{\partial \xi_\parallel} - \nabla \cdot \frac{\partial L}{\partial \nabla \xi_\parallel} = 0$$

$$\cancel{\gamma P_0 (\nabla \cdot \underset{\sim}{b})(\nabla \cdot \underset{\sim}{\xi})} - \underbrace{\nabla \cdot \left[ \gamma P_0 (\nabla \cdot \underset{\sim}{\xi}) \underset{\sim}{b} \right]} = 0$$

$$\cancel{\gamma P_0 (\nabla \cdot \underset{\sim}{b})} (\nabla \cdot \underset{\sim}{\xi})$$

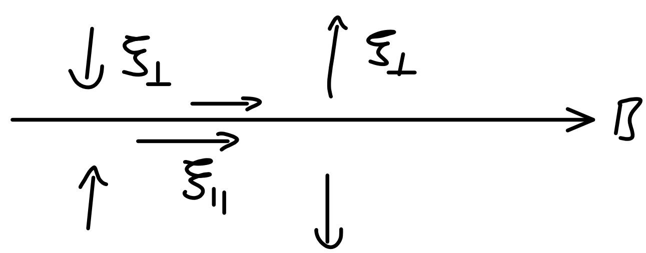
$$+ (\nabla \cdot \underset{\sim}{\xi}) \cancel{\underset{\sim}{b} \cdot \nabla (\gamma P_0)} \qquad \underset{\sim}{b} \cdot \nabla p_0 = 0$$

$$+ \gamma P_0 (\underset{\sim}{b} \cdot \nabla)(\nabla \cdot \underset{\sim}{\xi}) \qquad \underset{\sim}{J}_0 \times \underset{\sim}{B}_0 = \nabla P_0$$

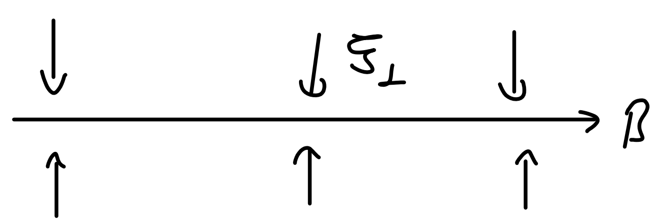
$$\boxed{(b \cdot \nabla)(\nabla \cdot \underset{\sim}{\xi}) = 0}$$

① Incompressible $(\underset{\sim}{b} \cdot \nabla) \neq 0$ variation along field lines

$\Rightarrow \nabla \cdot \underset{\sim}{\xi} = 0$

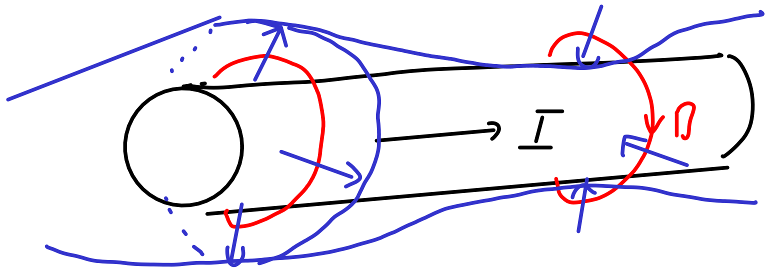



② Singular surfaces $(\underset{\sim}{b} \cdot \nabla) = 0$ constant along field lines




$\nabla \cdot \underset{\sim}{\xi} \neq 0$

e.g. z-pinch




m=0 "sausage" instability

stabilised by compression