Delta-W Pinch Stability

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$$\delta W = \delta W_p + \delta W_s + \delta W_v$$

incompressible

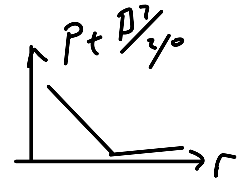
$$\delta W_p = \frac{1}{2}\int d\underline{x} \left[\frac{|\delta B_\perp|^2}{\mu_0} + \frac{B_0^2}{2\mu_0}|\nabla\cdot\underline{\xi}_\perp + \underline{\xi}_\perp\cdot\underline{\kappa}|^2 + \cancel{\gamma P_0 |\nabla\cdot\underline{\xi}|^2} \right.$$

$$\left. - 2(\underline{\xi}\cdot\nabla P_0)(\kappa\cdot\underline{\xi}_\perp) - \cancel{\delta\underline{B}\cdot(\underline{\xi}\times\underline{b})J_\parallel} \right]$$

no $J_\parallel$

$$\delta W_s = \frac{1}{2}\oint \left[\underbrace{\gamma P_0 \nabla\cdot\underline{\xi} + \xi_\perp\cdot\nabla P_0}_{\delta P} - \frac{\underline{B}\cdot\delta\underline{B}}{\mu_0} \right] \underline{\xi}^*\cdot d\underline{s}$$

$$= \frac{1}{2}\oint [[\nabla(\frac{B^2}{2\mu_0} + P)]]\, \underline{\xi}\cdot d\underline{s}$$



jump in gradient

→ vanishes if plasma smooth across plasma edge

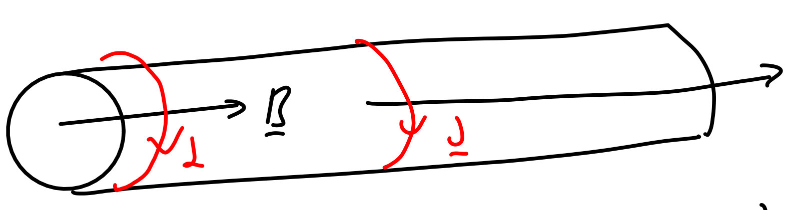
$$\delta W_p = \frac{1}{2}\int d\underline{x}\, \frac{|\delta B|^2}{\mu_0} > 0$$

Theta pinch

$$\underline{J} = \frac{b\times\nabla P}{qnB^2} \qquad \underline{J}\perp\underline{B}$$

$$J_\parallel = 0$$

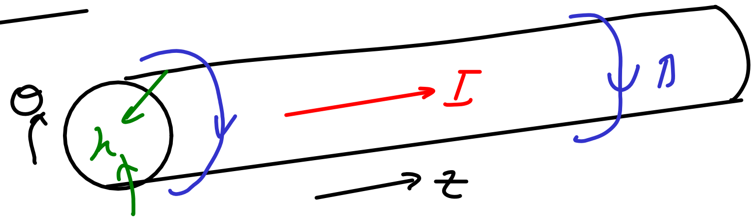
no kink



B field straight ⇒ $\underline{\kappa} = 0$

$\Rightarrow$ theta pinch is MHD stable

z pinch



$\underset{\sim}{J} \perp \underset{\sim}{B}$

$\Rightarrow J_{\parallel} = 0$

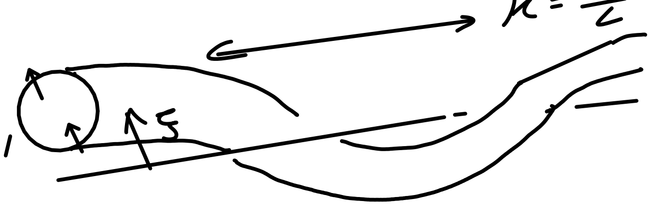
no kink

$\nabla \cdot \underset{\sim}{\xi} = 0$ except $\underset{\sim}{b} \cdot \nabla = 0$

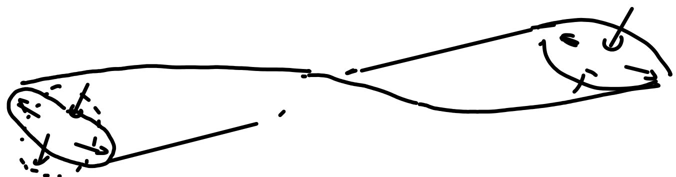
$k = \frac{2\pi}{L}$

mode number

$m=1$



$m=2$



$$\underset{\sim}{\xi}(r,\theta,z,t) = \hat{\xi}(r,t)\, e^{im\theta + ikz}$$

Fourier modes

$$\nabla \cdot \underset{\sim}{\xi} = \frac{1}{r}\frac{\partial}{\partial r}(r\xi_r) + \frac{1}{r}\frac{\partial \xi_\theta}{\partial \theta} + \frac{\partial \xi_z}{\partial z} = 0$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r\hat{\xi}_r) + \frac{im}{r}\hat{\xi}_\theta + ik\hat{\xi}_z = 0$$

$$\xi_\theta = \xi_{\parallel} = \frac{i}{m}\left[\frac{\partial}{\partial r}(r\hat{\xi}_r) + ikr\,\xi_z\right] \quad \text{if } m \neq 0$$

$\Rightarrow$ can set $\underline{\nabla \cdot \xi = 0}$

$$\delta \underset{\sim}{B} = \nabla \times (\underset{\sim}{\xi} \times \underset{\sim}{B}) = \xi(\cancel{\nabla \cdot \underset{\sim}{B}}) - B(\cancel{\nabla \cdot \xi}) + (\underset{\sim}{B} \cdot \nabla)\xi - (\xi \cdot \nabla)B$$

$$(\underline{B}\cdot\nabla)\underline{\xi} = \frac{B_\theta}{r}\frac{\partial}{\partial\theta}\underline{\xi} \qquad \underbrace{\underline{\xi}\cdot\nabla\underline{B}}_{\xi_r \frac{dB_\theta}{dr}\hat{\underline{\theta}}} \qquad \underline{B} \text{ constant in } z, \theta$$

$$\delta\underline{B} = \frac{B_\theta}{r}\frac{\partial}{\partial\theta}\left(\xi_r\hat{\underline{r}} + \xi_\theta\hat{\underline{\theta}} + \xi_z\hat{\underline{z}}\right) - \xi_r\frac{dB_\theta}{dr}\hat{\underline{\theta}}$$

$$\delta B_\perp = \frac{B_\theta}{r}\frac{\partial}{\partial\theta}\left(\xi_r\hat{r} + \xi_z\hat{\underline{z}}\right) = \frac{imB_\theta}{r}\left(\hat{\xi}_r\hat{\underline{r}} + \hat{\xi}_z\hat{\underline{z}}\right)$$

$$\nabla\cdot\underline{\xi}_\perp + 2\underline{\xi}_\perp\cdot\underline{\kappa} = \underbrace{\frac{1}{r}\frac{\partial}{\partial r}(r\xi_r)}_{} + ik\xi_z \underbrace{- \frac{2\xi_r}{r}}_{}$$

$$\underline{\kappa} = -\frac{1}{r}\hat{\underline{r}}$$

$$= r\frac{\partial}{\partial r}\left(\frac{\xi_r}{r}\right) + ik\xi_z$$

$$(\underline{\xi}\cdot\nabla P_0)(\underline{\kappa}\cdot\underline{\xi}_\perp^*) = -\frac{|\hat{\xi}_r|^2}{r}\frac{dP_0}{dr}$$

$$\delta W_p = \frac{1}{2}\int d\underline{x}\left[\frac{m^2B_\theta^2}{\mu_0 r^2}\left(|\xi_r|^2 + |\xi_z|^2\right) + \frac{B_\theta^2}{\mu_0}\left|r\frac{d}{dr}\left(\frac{\hat{\xi}_r}{r}\right) + ik\xi_z\right|^2 + \frac{2|\xi_r|^2}{r}\frac{dP_0}{dr}\right]$$

$$\frac{|\delta B_\perp|^2}{\mu_0}$$

Magnetic compression

$$
\begin{aligned}
\delta W_p = \frac{1}{2}\int dx \Big\{ & \underline{\left(\frac{m^2 B_\theta^2}{\mu_0 r^2} + \frac{2}{r}\frac{dP_0}{dr}\right)} |\xi_r|^2 \\
& + \cancel{\frac{1}{(m^2/r^2 + k^2)}\left|\left(\frac{m^2}{r^2}+k^2\right)\xi_z - ikr\frac{d}{dr}\left(\frac{\xi_r}{r}\right)\right|^2} \\
& + \frac{m^2}{m^2 + k^2 r^2}\left| r\frac{d}{dr}\left(\frac{\xi_r}{r}\right)\right|^2 \Big\}
\end{aligned}
$$

choose ξ_z

only appearance

k large most unstable

choose $\xi_z = \dfrac{ikr^3}{m^2+k^2r^2}\dfrac{d}{dr}\left(\dfrac{\xi_r}{r}\right)$

$$\Rightarrow \boxed{\frac{m^2 B_\theta^2}{\mu_0 r^2} + \frac{2}{r}\frac{dP_0}{dr} > 0}$$

for stability

(note $\dfrac{dP}{dr} < 0$)

$\Rightarrow$ limit to $\left|\dfrac{dP}{dr}\right|$ Kadomtsev 1966