The Ballooning Equation

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- Ordering for localised modes
- Lowest order delta-W
- Euler equation for perturbation

$$\delta W = \frac{1}{2}\int d\underset{\sim}{x}\Big[\underbrace{\frac{|\delta B_\perp|^2}{\mu_0}}_{1/\varepsilon^2} + \frac{1}{\mu_0}B_0^2\,\underbrace{|\nabla\cdot\xi_\perp + \cancel{2\xi_\perp\cdot\kappa}|^2}_{1/\varepsilon^2} + \underbrace{\gamma p|\nabla\cdot\underset{\sim}{\xi}|^2}_{1/\varepsilon^2}$$

$$-\underbrace{2(\underset{\sim}{\xi}\cdot\nabla p_0)(\kappa\cdot\xi_\perp)}_{\sim 1} - \underbrace{\cancel{\delta \underset{\sim}{B}\cdot(\xi_\perp\times b)J_\parallel}}_{\sim 1}\Big]$$

slowly varying

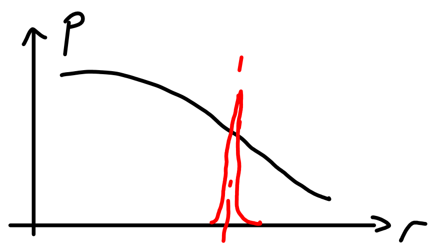
$\to 0$ at surface

$\delta B_\perp = (b\times\delta B)\times b \qquad \delta \underset{\sim}{B} = \nabla\times(\underset{\sim}{\xi}\times \underset{\sim}{B}) \sim \frac{1}{\varepsilon}$

$\frac{1}{\xi}|\nabla\xi| \sim \frac{1}{\varepsilon} \qquad \varepsilon << 1$

equilibrium

$\frac{1}{B}|\nabla B| \sim 1$



$$\frac{1}{\varepsilon^2}:\quad \delta W^{(-2)} = \frac{1}{2}\int d\underset{\sim}{x}\Big[\frac{|\delta B_\perp|^2}{\mu_0} + \frac{B_0^2}{\mu_0}|\nabla\cdot\xi_\perp|^2 + \underbrace{\cancel{\gamma p_0|\nabla\cdot\underset{\sim}{\xi}|^2}}_{\text{choose } \xi_\parallel \to 0}\Big]$$

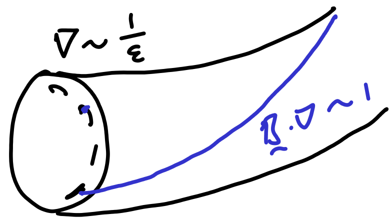
$$\xi_\perp^{(0)} = \frac{\underset{\sim}{B}\times\nabla\phi^{(1)}}{B^2}$$

$$\underset{\sim}{\xi} = \underbrace{\underset{\sim}{\xi}^{(0)}}_{\sim 1} + \underbrace{\underset{\sim}{\xi}^{(1)}}_{\sim\varepsilon} + \underbrace{\xi^{(2)}}_{\sim\varepsilon^2} + \cdots$$

$$|\delta B_\perp|^2 = \frac{1}{B^2}\Big|\overset{1/\varepsilon}{\nabla}\big(\underbrace{\underset{\sim}{B}\cdot\nabla}_{\sim 1}\overset{\sim\varepsilon}{\phi^{(1)}}\big)\Big|^2$$

$|\delta \underline{B}_\perp|^2 \sim 1$ if $\underline{\underline{B}\cdot\nabla\phi \sim 1}$

Perturbation gradients along magnetic field on same scale as equilibrium



$$\varepsilon^0 : \quad \delta W^{(0)} = \frac{1}{2}\int \frac{|\delta B_\perp|^2}{\mu_0} + \underbrace{\frac{B^2}{\mu_0}|\overset{1/\varepsilon}{\nabla}\cdot \overset{\varepsilon}{\xi_\perp}|^2}_{\text{choose } \xi^{(1)}} - 2(\xi_\perp\cdot\nabla p_0)(\underline{\kappa}\cdot\xi_\perp)\, dV$$

$$\delta W^{(0)} = \frac{1}{2}\int d\underline{x}\Big[\underbrace{\frac{1}{\mu_0 B^2}\Big|\underset{\sim 1/\varepsilon}{\nabla}(\underset{\sim 1}{\underline{B}\cdot\nabla\phi})\Big|^2 - 2\Big[\frac{\nabla p\times \underline{B}}{B^2}\cdot\nabla\phi\Big]\Big[\frac{\underline{\kappa}\times\underline{B}}{B^2}\cdot\nabla\phi\Big]}_{L}\Big]$$

Minimise using calculus of variations

$$\frac{\delta W^{(0)}}{\delta\phi} = \underset{0}{\cancel{\frac{\partial L}{\partial\phi}}} - \nabla\cdot\frac{\partial L}{\partial\nabla\phi} = 0$$

$$\frac{\delta}{\delta\nabla\phi}\int|\nabla(\underline{B}\cdot\nabla\phi)|^2 d\underline{x} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int|\nabla(\underline{B}\cdot[\nabla\phi+\varepsilon\underline{\delta}])|^2 - |\nabla(\underline{B}\cdot\nabla\phi)|^2 d\underline{x}$$

$$= \int 2\nabla(\underline{B}\cdot\underline{\delta})\cdot\nabla(\underline{B}\cdot\nabla\phi)\, d\underline{x}$$

$$= -2\int \underline{B}\cdot\underline{\delta}\,\nabla^2(\underline{B}\cdot\nabla\phi)\, d\underline{x} = -2\underline{B}\,\nabla^2(\underline{B}\cdot\nabla\phi)$$

$$\underset{\sim}{B} \cdot \nabla \left[\frac{1}{\mu_0 B} \nabla^2 (\underset{\sim}{B} \cdot \nabla \phi) \right] + 2 \frac{\underset{\sim}{\kappa} \times \underset{\sim}{B}}{B^2} \cdot \nabla \left[\frac{\nabla p \times \underset{\sim}{B}}{B^2} \cdot \nabla \phi \right] = 0$$

Single scalar field ϕ

$\Rightarrow$ Can be simplified using the Ballooning transform