Magnetic Helicity

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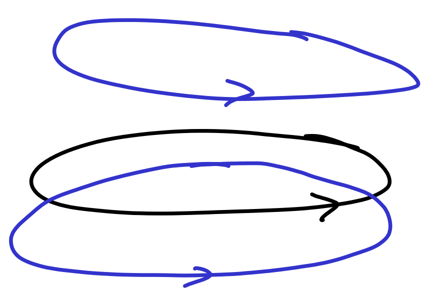
$$H = \int_V \underset{\sim}{B} \cdot \underline{A}\, dV \qquad \underset{\sim}{B} = \nabla \times \underline{A}$$

H is conserved in ideal MHD
approximately conserved in non-ideal MHD

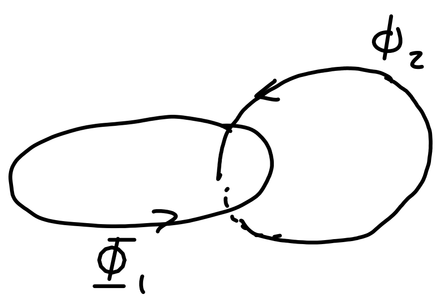
Note: H is a gauge dependent quantity

So $\underset{\sim}{A} \cdot \underset{\sim}{B}$ is not a helicity density

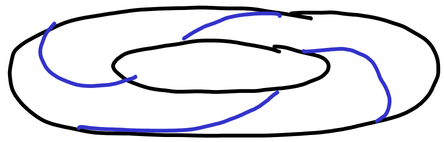
Examples



$H = 0$ field lines not linked



$H = \pm \Phi_1 \Phi_2\ 2$



$H \neq 0$

twisted field lines

Conserved in ideal MHD

$$H = \int \underset{\sim}{A} \cdot \underset{\sim}{B} \, dV \qquad \frac{dH}{dt} = \int \frac{\partial \underset{\sim}{A}}{\partial t} \cdot \underset{\sim}{B} \, dV + \int \underset{\sim}{A} \cdot \frac{\partial \underset{\sim}{B}}{\partial t} \, dV$$

$$\underset{\sim}{E} = -\nabla \phi - \frac{\partial A}{\partial t} \qquad \frac{\partial B}{\partial t} = -\nabla \times E$$

$$\frac{dH}{dt} = \int \left(-\underset{\sim}{E} - \underline{\nabla \phi}\right) \cdot \underset{\sim}{B} \, dV - \int \underbrace{\underset{\sim}{A} \cdot (\nabla \times \underset{\sim}{E})} \, dV$$

$$\nabla \phi \cdot \underset{\sim}{B} = \nabla \cdot (\phi \underset{\sim}{B}) \qquad \nabla \cdot (A \times \underset{\sim}{E}) = (\nabla \times A) \cdot E - A \cdot (\nabla \times \underset{\sim}{E})$$

$$\frac{dH}{dt} = -\int \underset{\sim}{E} \cdot \underset{\sim}{B} \, dV - \int \nabla \cdot (\phi \underset{\sim}{B}) \, dV + \int \nabla \cdot (\underset{\sim}{A} \times \underset{\sim}{E}) \, dV - \int \underbrace{(\nabla \times \underset{\sim}{A})}_{B} \cdot \underset{\sim}{E} \, dV$$

$$= -2 \int \underset{\sim}{E} \cdot \underset{\sim}{B} \, dV + \oint_S \left(\underset{\sim}{A} \times \underset{\sim}{E} - \underset{\sim}{B} \phi\right) \cdot d\underset{\sim}{S}$$

ideal MHD $\quad \underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B} = 0 \qquad \underset{\sim}{E} \cdot \underset{\sim}{B} = 0$

$\underset{\sim}{B} \cdot d\underset{\sim}{S} = 0$

$\underset{\sim}{E} \cdot d\underset{\sim}{S} = 0$ or $\underset{\sim}{A} \cdot d\underset{\sim}{S} = 0$

$\Rightarrow$ H is constant in time

Cross-Helicity

$$H_c = \int \underset{\sim}{u} \times \underset{\sim}{B} \, dV$$

in ideal MHD

$$\frac{dH_c}{dt} = \oint \left[\left(\frac{1}{2} u^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \right) \underset{\sim}{B} - \underset{\sim}{u} \times \left(\underset{\sim}{u} \times \underset{\sim}{B} \right) \right] \cdot d\underset{\sim}{S}$$

$\underset{\sim}{B} \cdot d\underset{\sim}{S} = 0$ and $\underset{\sim}{u} \cdot dS = 0$

$\Rightarrow H_c$ conserved