Woltjer's Theorem (1958)

Contents

- Magnetic helicity
- Minimisation of magnetic energy
- Force free field

Minimise $\quad W = \int_V B^2/2\mu_0 \, dV \qquad \underset{\sim}{B} = \nabla \times \underline{A}$

Constraint $\quad H = \int_V \underline{A} \cdot \underset{\sim}{B} \, dV$

$$\Rightarrow \quad W' = \int_V \left[(\nabla \times \underline{A})^2 + \lambda \underset{\sim}{A} \cdot \underset{\sim}{B} \right] dV$$

$$\delta W' = W'(\underset{\sim}{A} + \delta \underset{\sim}{A}) - W'(\underline{A})$$

$\delta \underset{\sim}{A}$ small $\rightarrow$ neglect δA^2 terms

$$W' = \int \Big[\underbrace{2(\nabla \times A)(\nabla \times \delta A)}_{\nabla \cdot (\delta \underline{A} \times (\nabla \times A)) \, + \, \delta \underline{A} \cdot \nabla \times \underbrace{\nabla \times \underline{A}}_{B}} + \underbrace{\lambda \underline{A} \cdot \nabla \times \delta \underline{A}}_{\nabla \cdot (\delta \underline{A} \times \underline{A}) \, + \, \delta \underline{A} \cdot (\nabla \times \underline{A})} + \lambda \delta \underset{\sim}{A} \cdot \nabla \times \underline{A} \Big] dV$$

$$W' = \int \Big[\cancel{2\nabla\cdot(\delta \underset{\sim}{A} \times (\nabla\times \underline{A}))} + 2\,\delta \underset{\sim}{A} \cdot \nabla\times \underset{\sim}{B}$$

$$+ \cancel{\nabla\cdot(\delta \underset{\sim}{A} \times \underline{A})} + 2\lambda\, \delta \underset{\sim}{A} \cdot \nabla\times \underline{A} \Big]\, dV$$

$\delta \underset{\sim}{A} = 0$ at boundary

$$W' = 2\int \underbrace{(\nabla\times \underset{\sim}{B} + \lambda \underset{\sim}{B})}_{=0} \cdot \delta \underset{\sim}{A}\, dV = 0$$

(Minimum)

$$\boxed{\nabla\times \underset{\sim}{B} + \lambda \underset{\sim}{B} = 0}$$

Force-free field