Conservation properties

Contents

- Directly conserved (fluxes)
- Indirectly conserved (integration by parts)

1) Directly conserved

e.g. density, momentum, sometimes energy

evolving quantity

$$\frac{\partial f}{\partial t} + \nabla \cdot (\underset{\sim}{F}) = S$$

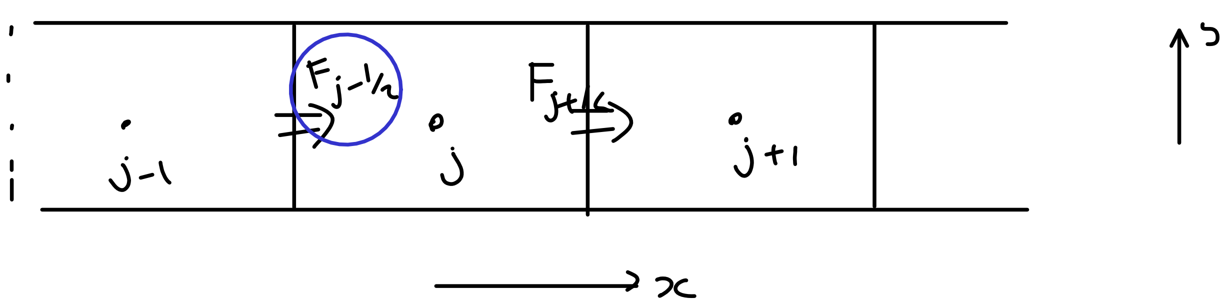
flux

source

e.g. $f = \rho$ mass density $\quad \underset{\sim}{F} = \rho \underset{\sim}{u}$

Integrate over a volume

$$\frac{\partial}{\partial t}\int f \, dV + \underbrace{\int \nabla \cdot \underset{\sim}{F} \, dV}_{\oint \underset{\sim}{F} \cdot d\underset{\sim}{S}} = \int S \, dV$$



$$\frac{\partial f_j}{\partial t} = \frac{F_{j-1/2} - F_{j+1/2}}{\Delta V} \qquad F_{j+1/2} = \rho_{j+1/2} u_{j+1/2} \Delta y$$

$\Delta V = \Delta x \Delta y$

$$\frac{\partial f_{j-1}}{\partial t} = \frac{F_{j-3/2} - F_{j-1/2}}{\Delta V_{j-1}}$$

By writing equations in flux form conservation of $\int f \, dv$ is ensured even if $\underset{\sim}{F}$ is approximated.

Application to MHD

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho \underset{\sim}{u} \\ w \end{pmatrix} + \nabla \cdot \begin{pmatrix} \rho \underset{\sim}{u} \\ \rho \underset{\sim}{u}\underset{\sim}{u} + \left(p + \frac{B^2}{2\mu_0}\right)\mathbb{I} - \frac{\underset{\sim}{B}\underset{\sim}{B}}{2\mu_0} \\ \left(\frac{\rho \underset{\sim}{u}^2}{2} + \frac{\gamma}{\gamma-1} p\right)\underset{\sim}{u} + \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{\mu_0} \end{pmatrix} = 0$$

Ideal MHD in conservation form

$\Rightarrow$ can ensure that $\int \rho \, dv$ conserved

$\int \rho \underset{\sim}{u} \, dv$

$\int w \, dv$

Indirectly Conservation

Combination of several equations

$\rightarrow$ Integral relations

Example $\quad \dfrac{\partial p}{\partial t} = -p\nabla\cdot\underline{u} \qquad \dfrac{\partial \underline{u}}{\partial t} = -\nabla p$

Sound wave

$$\boxed{\nabla\cdot(p\underline{u}) = p\nabla\cdot\underline{u} + \underline{u}\cdot\nabla p}$$

Depends on this

Conserved energy

$$\underbrace{\underline{u}\cdot\frac{\partial \underline{u}}{\partial t}}_{\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right)} = -\underline{u}\cdot\nabla p = p\nabla\cdot\underline{u} - \nabla\cdot(p\underline{u})$$

$$\frac{\partial p}{\partial t} = -p\nabla\cdot\underline{u}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) = p\nabla\cdot\underline{u} - \underbrace{\nabla\cdot(p\underline{u})}_{\text{boundary}}$$

$$\int_\Omega \left(p + \frac{1}{2}u^2\right) dV = \int_\Omega -\nabla\cdot(p\underline{u})\, dV$$

conserved quantity

$$= -\oint_{\partial\Omega} p\underline{u}\cdot d\underline{s}$$

flux of energy through boundary

Conservation depends on properties of $\nabla\cdot()$ and ∇ operators $\rightarrow$ choose carefully!

e.g. central difference

Summation by parts (SBP)