Cold plasma wave dispersion

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$$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J} + \frac{1}{c^2}\frac{\partial \underset{\sim}{E}}{\partial t}$$

$$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$$

$$\sim e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)} \qquad \nabla \longrightarrow i\underset{\sim}{k}$$

$$\frac{\partial}{\partial t} \longrightarrow -i\omega$$

$$\underset{\sim}{k} \times \underset{\sim}{B} = -i\mu_0 \underset{\sim}{J} - \frac{\omega}{c^2}\underset{\sim}{E}$$

$$\underset{\sim}{k} \times \underset{\sim}{E} = \omega \underset{\sim}{B}$$

$$\underbrace{\underset{\sim}{k} \times (\underset{\sim}{k} \times \underset{\sim}{E})}_{\underset{\sim}{k}(\underset{\sim}{k}\cdot\underset{\sim}{E}) - k^2\underset{\sim}{E}} = -i\omega\mu_0 \underset{\sim}{J} - \frac{\omega^2}{c^2}\underset{\sim}{E}$$

plasma response

$$\underset{\sim}{J} = \underset{=}{\sigma} \cdot \underset{\sim}{E} \qquad \underset{=}{\sigma} \text{ conductivity tensor}$$

$$\underbrace{\left[\underset{\sim}{k}\,\underset{\sim}{k} - k^2 + i\omega\mu_0 \underline{\underline{\sigma}} + \frac{\omega^2}{c^2}\right]}_{\text{tensor (3×3 matrix)}} \cdot \underset{\sim}{E} = 0$$

Determinant $= 0$ allows $\underset{\sim}{E} \neq 0$

<u>Cold plasma</u> $\quad p_e = p_i = 0$

Momentum equations for ions, electrons

$$m_i n \frac{\partial \underset{\sim}{u}_i}{\partial t} = \underset{\sim}{J} \times \underset{\sim}{B}_0 \qquad\qquad e^{i(\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega t)}$$

$$\underset{\sim}{E} + \underset{\sim}{u}_i \times \underset{\sim}{B} = \frac{\underset{\sim}{J}\times B_0}{en} + \frac{m_e}{ne^2}\frac{\partial \underset{\sim}{J}}{\partial t}$$

$$-i\omega m_i n \underset{\sim}{u}_i = \underset{\sim}{J} \times B_0$$

$$\underset{\sim}{E} + \underset{\sim}{u}_i \times B_0 = \frac{\underset{\sim}{J}\times B_0}{en_0} - \frac{m_e}{ne^2} i\omega \underset{\sim}{J}$$

$$\underset{\sim}{E} + \underbrace{\frac{i}{\omega m_i n}}_{\frac{1}{\omega \Pi_e^2}\frac{e^2}{m_e m_i \varepsilon_0}} \underbrace{(\underset{\sim}{J}\times \underset{\sim}{B}_0)\times \underset{\sim}{B}_0}_{\underset{\sim}{J}_\perp B_0^2} = \frac{\underset{\sim}{J}\times \underset{\sim}{B}_0}{en_0} - i\omega \underbrace{\frac{m_e}{e^2 n}}_{\frac{1}{\Pi_e^2 \varepsilon_0}} \underset{\sim}{J}$$

$$\Omega_e \Omega_i \qquad\qquad \Pi_e = \sqrt{\frac{ne^2}{\varepsilon_0 m_e}}$$

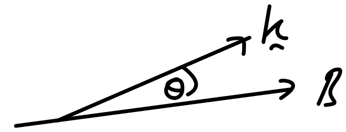
$$\Omega_e = \frac{-eB_0}{m_e} \qquad \Omega_i = \frac{eB_0}{m_i}$$

$$i\omega\varepsilon_0 \underset{\sim}{E} = \frac{1}{\Pi_e^2}\left[\Omega_e\Omega_i \underset{\sim}{J}_\perp - i\omega\Omega_e \underset{\sim}{J}\times \underset{\sim}{b} + \omega^2 \underset{\sim}{J}\right]$$

Relationship between $\underset{\sim}{E}$ and $\underset{\sim}{J}$ $\rightarrow$ $\underset{=}{\sigma}$

choose z along $\underset{\sim}{B}$

$$\underset{\sim}{k} = \begin{pmatrix} k_x \\ 0 \\ k_z \end{pmatrix} = \begin{pmatrix} k_\perp \\ 0 \\ k_\parallel \end{pmatrix} = \begin{pmatrix} k\sin\theta \\ 0 \\ k\cos\theta \end{pmatrix}$$



Result in

$$\underbrace{\begin{pmatrix} S - N_\parallel^2 & -iD & N_\parallel N_\perp \\ iD & S - N^2 & 0 \\ N_\parallel N_\perp & 0 & P - N_\perp^2 \end{pmatrix}}_{\text{Determinant} = 0} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

$$\underset{\sim}{N} = \frac{c\underset{\sim}{k}}{\omega} \qquad N_\parallel = \frac{ck_\parallel}{\omega} \qquad N_\perp = \frac{ck_\perp}{\omega}$$

$N^2 > 0$ propagating (oscillating)

$N^2 < 0$ decaying wave (evanescent)

$N^2 \to 0$ reflect

$N^2 \to \infty$ absorption (resonance)

$$S = \frac{R+L}{2} \qquad D = \frac{R-L}{2}$$

$$R \simeq 1 - \frac{\Pi_e^2}{(\omega + \Omega_e)(\omega + \Omega_i)} \qquad L \simeq 1 - \frac{\Pi_e^2}{(\omega - \Omega_e)(\omega - \Omega_i)}$$

Right handed note $\Omega_e < 0$ Left handed

$$P = 1 - \frac{\Pi_e^2}{\omega^2} \qquad \text{plasma frequency}$$

Parallel waves

$$\underset{\sim}{k} = k_{\parallel} \underset{\sim}{b} \qquad k_{\perp} = k_x = 0 \qquad N_{\perp} = 0$$

$$\underbrace{\begin{pmatrix} S - N^2 & -iD & 0 \\ iD & S - N^2 & 0 \\ 0 & 0 & P \end{pmatrix}}_{\text{Determinant}} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

$$(S - N^2)^2 P - (-iD)(iD) P = 0$$

$$\left[(S - N^2)^2 - D^2 \right] P = 0$$

$P = 0 \qquad \omega^2 = \Pi_e^2$ Langmuir wave

no group velocity

$$(S - N^2)^2 - D^2 = 0$$

$$\left(\frac{R+L}{2} - N^2 \right)^2 = \left(\frac{R-L}{2} \right)^2$$

$N^2 = R$ right handed

$N^2 = L$ left handed

whistler waves