Implicit methods

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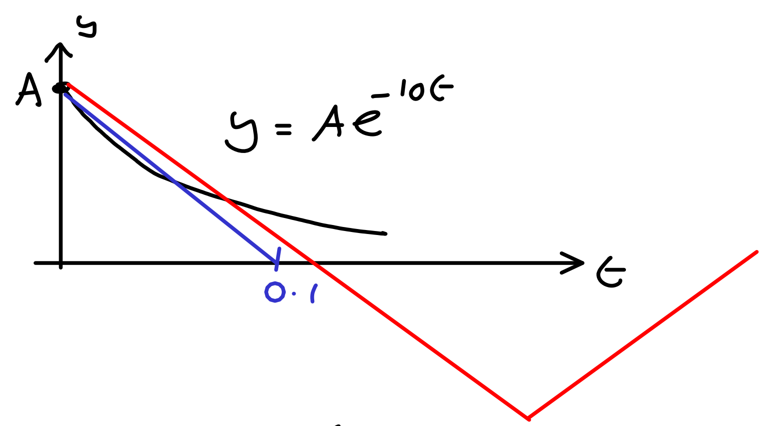
Aim: Overcome timestep restrictions due to fast processes

e.g. Fast magnetosonic waves, or heat conduction



Example

$$\frac{dy}{dt} = -10y$$

Euler (explicit)

$$\frac{y^{n+1} - y^n}{\Delta t} = -10 y^n \qquad y^{n+1} = y^n(1 - 10\Delta t)$$

$$\text{if } \Delta t = 0.1 \Rightarrow y^{n+1} = 0 \quad *$$

$$= 0.2 \Rightarrow y^{n+1} = -y^n \quad *$$

$$> 0.2 \Rightarrow |y^{n+1}| > |y^n|$$

unstable

Timestep restriction

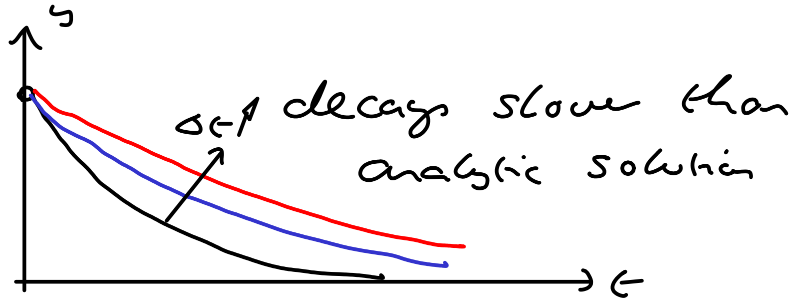
Implicit   Backward Euler

$$\frac{y^{n+1} - y^n}{\Delta t} = -10 y^{n+1} \qquad y^{n+1}(1 + 10\Delta t) = y^n$$

$$y^{n+1} = y^n / (1 + 100\Delta t) > 0$$

always stable

A-stable method, stable for e^{-t}



Oscillation

$$\frac{df}{dt} = g \qquad \frac{dg}{dt} = -f \qquad f = f_0 e^{it}$$

Backward Euler

$$\frac{f^{n+1} - f^n}{\Delta t} = g^{n+1} \qquad \frac{g^{n+1} - g^n}{\Delta t} = -f^{n+1}$$

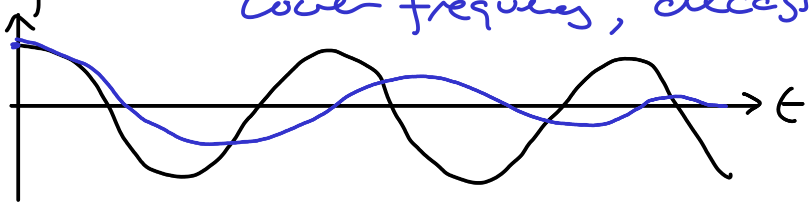
$$f^{n+1} - \Delta t\, g^{n+1} = f^n \qquad g^{n+1} + \Delta t\, f^{n+1} = g^n$$

$$\begin{pmatrix} 1 & -\Delta t \\ \Delta t & 1 \end{pmatrix} \begin{pmatrix} f^{n+1} \\ g^{n+1} \end{pmatrix} = \begin{pmatrix} f^n \\ g^n \end{pmatrix}$$

$$\begin{pmatrix} f^{n+1} \\ g^{n+1} \end{pmatrix} = \underbrace{\frac{1}{1 + \Delta t^2}}_{\text{damping}} \underbrace{\begin{pmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{pmatrix} \begin{pmatrix} f^n \\ g^n \end{pmatrix}}_{\text{Forward Euler}}$$

$\Delta t \to \infty$ damps $\to 0$

Lower frequency, decays in time



Implicit methods tend to damp oscillations and waves, particularly when unresolved.

Semi-implicit (IMEX)

- fast processes - not resolved ← IMPLICIT
- slow processes - resolve ← EXPLICIT

$$\frac{d\underset{\sim}{y}}{dt} = \underset{\substack{\uparrow \\ \text{fast}}}{F(\underset{\sim}{y})} + \underset{\substack{\uparrow \\ \text{slow}}}{G(\underset{\sim}{y})}$$

$$\frac{y^{n+1} - y^n}{\Delta t} = F(y^{n+1}) + G(y^n)$$

$$\underbrace{y^{n+1} - \Delta t F(y^{n+1})}_{\text{nonlinear}} = y^n + \Delta t G(y^n)$$

→ Newton, predictor-corrector