Poisson brackets in Reduced MHD

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$$\underset{\sim}{V}_E \cdot \nabla f = \frac{1}{B}\, \underset{\sim}{b} \times \nabla \phi \cdot \nabla f$$

in field-aligned coordinates $\quad \underset{\sim}{B} = \nabla x \times \nabla y$

$$(\underset{\sim}{b} \times \nabla \phi) \cdot \nabla f = (\nabla \phi \times \nabla f) \cdot \underset{\sim}{b}$$

$$\nabla \phi = \nabla x \frac{\partial \phi}{\partial x} + \nabla y \frac{\partial \phi}{\partial y} + \nabla z \frac{\partial \phi}{\partial z}$$

$$\begin{aligned} B\, \underset{\sim}{V}_E \cdot \nabla f = & \left(\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial f}{\partial x} \right) \underbrace{(\nabla x \times \nabla y)}_{B} \cdot \underset{\sim}{b} \\ & + \left(\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial z} - \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial x} \right) \underbrace{(\nabla x \times \nabla z)}_{\frac{1}{J}\underset{\sim}{e}_y} \cdot \underset{\sim}{b} \\ & + \left(\frac{\partial \phi}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial y} \right) \underbrace{(\nabla y \times \nabla z)}_{\frac{1}{J}e_x} \cdot \underset{\sim}{b} \end{aligned}$$

$$\underset{\sim}{V}_E \cdot \nabla f \simeq \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial f}{\partial x}$$

Coordinates ~ orthogonal

$$= [\phi, f]$$

poisson bracket

antisymmetric operator perpendicular to $\underset{\sim}{B}$

$$\frac{\partial n}{\partial t} + [\phi, n] = 0$$ incompressible flow

$$\frac{1}{B}\underset{\sim}{b}\times\nabla\phi\cdot\nabla f = \nabla\cdot\left(\frac{\underset{\sim}{b}\times\nabla\phi}{B}f\right) - f\,\cancel{\nabla\cdot\left(\frac{\underset{\sim}{b}\times\nabla\phi}{B}\right)}$$

$$\frac{\partial}{\partial t}\int n\,dV + \int\nabla\cdot\left(\frac{\underset{\sim}{b}\times\nabla\phi}{B}f\right)dV = 0$$

conservation

$$\int[\phi, n]\,dV = 0$$ if no boundary flux

numerically $\sum_i [\phi, n]_i\,\Delta V_i = 0$

discrete form of conservation law

$$\frac{\partial\omega}{\partial t} + [\omega, \phi] = 0 \qquad \omega = \nabla^2\phi$$

$$\underbrace{\omega\frac{\partial\omega}{\partial t}}_{\frac{1}{2}\frac{\partial}{\partial t}(\omega^2)} + \underbrace{\omega[\omega, \phi]}_{\frac{1}{2}\nabla\cdot\left(\omega^2\frac{\underset{\sim}{b}\times\nabla\phi}{B}\right)} = 0$$ (incompressible)

$\Rightarrow$ ω^2 conserved (enstrophy)

2D grid i, j

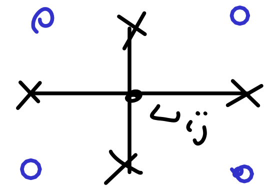
$$\frac{\partial}{\partial t}(\omega_{ij}^2) + \sum_{i,j} \omega_{ij}[\phi,\omega]_{ij} = 0 \qquad \text{discrete form}$$

should $= 0$ if $\frac{\partial}{\partial t} = 0$

2nd order central difference

$$\left.\frac{\partial \phi}{\partial x}\right|_{i,j} \simeq \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x}$$

$$[\phi,\omega] \simeq \frac{(\phi_{i+1,j} - \phi_{i-1,j})(\omega_{i,j+1} - \omega_{i,j-1})}{4\Delta x \Delta y} - \frac{(\phi_{i,j+1} - \phi_{i,j-1})(\omega_{i+1,j} - \omega_{i-1,j})}{4\Delta x \Delta y}$$



$$\omega_{ij}[\phi,\omega]_{ij} = \omega_{ij}(\phi_{i+1,j} - \phi_{i-1,j})\,\omega_{i,j+1} + \cdots$$

$$\omega_{i,j+1}[\phi,\omega]_{i,j+1} = \omega_{i,j+1}(\phi_{i+1,j+1} - \phi_{i-1,j+1})(-\omega_{ij}) + \cdots$$

$\Rightarrow$ Terms do not balance

$\Rightarrow$ breaks conservation of enstrophy

Arakawa (1966)

$\rightarrow$ conserves total n, ω and ω^2