Derivation of CGL MHD

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- Neglect heat fluxes => CGL

References:

- P.Hunana et al. arXiv:1901.09354 (2019)
- J.Freidberg "Ideal MHD" (2014, Cambridge)

Collisionless Vlasov equation

$$\frac{\partial f}{\partial t} + \underset{\sim}{v}\cdot\nabla f + \frac{q}{m}(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B})\cdot\underbrace{\nabla_v}_{\frac{\partial}{\partial \underset{\sim}{v}}} f = 0$$

$$f(t, \underset{\sim}{x}, \underset{\sim}{v})$$

Moment

$$n = \int f\, d\underset{\sim}{v}$$

$$n\underset{\sim}{u} = \int \underset{\sim}{v} f\, d\underset{\sim}{v}$$

particle speed relative to fluid

$$\underset{\sim}{w} = \underset{\sim}{v} - \underset{\sim}{u}$$

fluid

$$\underset{=}{P} = m\int(\underset{\sim}{v} - \underset{\sim}{u})(\underset{\sim}{v} - \underset{\sim}{u})\, f\, d\underset{\sim}{v}$$

Coordinate (kinetic)

rank 2 tensor (matrix)

note $\quad \underset{\sim}{a}\underset{\sim}{b} \quad a_i b_j$

$$\underline{\underline{\underline{q}}} = m \int (\underset{\sim}{v} - \underset{\sim}{u})(\underset{\sim}{v} - \underset{\sim}{u})(\underset{\sim}{v} - \underset{\sim}{u}) f \, d\underset{\sim}{v}$$

rank 3 tensor

Pressure tensor equation

$$\underbrace{\frac{\partial \underline{\underline{P}}}{\partial t}} + \nabla \cdot (\underset{\sim}{u} \underline{\underline{P}} + \underline{\underline{\underline{q}}}) + \Big[\underline{\underline{P}} \cdot \nabla \underset{\sim}{u} + \underbrace{\frac{q}{m} \underset{\sim}{B} \times \underline{\underline{P}}} \Big]^S = 0$$

matrix equation $\quad \underline{\underline{A}}^S = A + A^T$

$$\nabla \cdot (\underset{\sim}{u} \underline{\underline{P}}) = \partial_k (u_k P_{ij}) \qquad (\underset{\sim}{B} \times \underline{\underline{P}})_{ij} = \varepsilon_{ikl} B_k P_{lj}$$

$$\nabla \cdot \underline{\underline{\underline{q}}} = \partial_k q_{kij}$$

$$\underline{\underline{P}} \cdot \nabla \underset{\sim}{u} = P_{ik} \partial_k u_j$$

$$\frac{q}{m} \underset{\sim}{B} \times \underline{\underline{P}} \simeq \Omega \underline{\underline{P}} \qquad \frac{\partial \underline{\underline{P}}}{\partial t} \sim \omega \underline{\underline{P}}$$

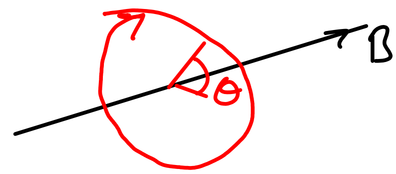
If in MHD limit $\omega \ll \Omega$

$\Rightarrow \frac{q}{m} \underset{\sim}{B} \times \underline{\underline{P}}$ must go to zero or it will dominate

(1) Ideal MHD $\quad \underline{\underline{P}} = p \underline{\underline{\mathbb{I}}}$ $\quad$ isotropic

$$\frac{q}{m} \underset{\sim}{B} \times \underline{\underline{P}} + \frac{q}{m} (\underset{\sim}{B} \times \underline{\underline{P}})^T = 0$$

② gyrotropic pressure – constant pressure around gyro-angle θ



$$\underline{\underline{P}} = P_\parallel \underline{b}\underline{b} + P_\perp (\underline{\underline{I}} - \underline{b}\underline{b}) \qquad \text{CGL MHD}$$

$$[+ \underline{\underline{\Pi}}]$$

↖ small FLR corrections

$$P_\parallel = \underline{\underline{P}} : \underline{b}\underline{b} \qquad P_\perp = \underline{\underline{P}} : \frac{(\underline{\underline{I}} - \underline{b}\underline{b})}{2}$$

$$\uparrow \qquad \uparrow$$

$$P_{ij} \qquad b_i b_j$$

$$\underline{\underline{P}} = \begin{pmatrix} P_\perp & & \\ & P_\perp & \\ & & P_\parallel \end{pmatrix}$$

note: $\underline{\underline{\Pi}} : \underline{b}\underline{b} = 0 \qquad \underline{\underline{\Pi}} : (\underline{\underline{I}} - \underline{b}\underline{b}) = 0$

$$\underline{\underline{\Pi}} : \underline{\underline{I}} = 0 \Rightarrow \mathrm{Tr}(\underline{\underline{\Pi}}) = 0$$

$$\boxed{\frac{\partial P_\parallel}{\partial t} + \nabla \cdot (P_\parallel \underline{u}) + 2 P_\parallel \underline{b} \cdot \nabla \underline{u} \cdot \underline{b}} + \underline{b} \cdot (\nabla \cdot \underline{\underline{q}}) \cdot \underline{b}$$

$$- \frac{2}{|B|} \underline{b} \cdot \underline{\underline{\Pi}} \cdot \frac{dB}{dt} + (\Pi \cdot \nabla \underline{u})^S : \underline{b}\underline{b} = 0$$

$$\boxed{\frac{\partial p_\perp}{\partial t} + \nabla\cdot(p_\perp \underline{u}) + p_\perp \nabla\cdot\underline{u} - p_\perp \underline{b}\cdot\nabla\underline{u}\cdot\underline{b}}$$

CGL

$\underline{\underline{q}} = 0 \quad \underline{\underline{\Pi}} = 0$

$$+ \frac{1}{2}\left[\mathrm{Tr}(\nabla\cdot\underline{\underline{q}}) - \underline{b}\cdot(\nabla\cdot\underline{\underline{q}})\cdot\underline{b}\right]$$

CGL $\rightarrow 0$

$$+ \frac{1}{2}\left(\mathrm{Tr}(\underline{\underline{\Pi}}\cdot\nabla\underline{u})^s + \frac{2}{|B|}\underline{b}\cdot\underline{\underline{\Pi}}\cdot\frac{dB}{dt} - (\underline{\underline{\Pi}}\cdot\nabla\underline{u})^s:\underline{b}\underline{b}\right) = 0$$