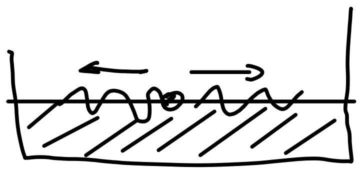
Linearisation and Waves

Concepts

- Expansion in small parameters
- Linearisation
- Wave dispersion relations



$$\rho = \rho_0 + \delta\rho$$
$$\mathbf{u} = \mathbf{u}_0 + \delta\mathbf{u}$$
$$P = P_0 + \delta P$$

$\uparrow$ Equilibrium $\quad \frac{\partial}{\partial t} \to 0$

① Density

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

$$\frac{\partial \delta\rho}{\partial t} + \nabla\cdot\left[(\rho_0 + \delta\rho)(\mathbf{u}_0 + \delta\mathbf{u})\right] = 0$$

$$\frac{\partial \delta\rho}{\partial t} + \nabla\cdot\Big[\overset{\text{Eq.}}{\cancel{\rho_0\mathbf{u}_0}} + \delta\rho\,\mathbf{u}_0 + \delta\mathbf{u}\,\rho_0 + \underbrace{\overset{\text{Nonlinear}}{\cancel{\delta\rho\,\delta\mathbf{u}}}}_{\text{Small}}\Big) = 0$$

$$\frac{\partial \delta\rho}{\partial t} + \nabla\cdot(\delta\rho\,\mathbf{u}_0 + \delta\mathbf{u}\,\rho_0) = 0$$

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② Momentum

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right] = -\nabla P$$

$$\rho_0\left[\frac{\partial \delta \underline{u}}{\partial t} + \delta \underline{u}\cdot\nabla \underline{u}_0 + \underline{u}_0\cdot\nabla\delta\underline{u}\right] = -\nabla\delta P$$

$$\delta\rho\left[\underline{u}_0\cdot\nabla\underline{u}_0\right]$$

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③ Pressure

$$\frac{\partial P}{\partial t} + \underline{u}\cdot\nabla P = -\gamma P\nabla\cdot\underline{u}$$

$$\underline{\frac{\partial \delta P}{\partial t} + u_0\cdot\nabla\delta P + \delta u\cdot\nabla P_0 = -\gamma P_0\nabla\cdot\delta\underline{u} - \gamma\delta P\nabla\cdot\underline{u}_0}$$

3 linear equations
→ perturbations

Example: Sound waves

uniform, stationary fluid

$\nabla\rho_0, \nabla P_0 = 0$ $\qquad \underline{u}_0 = 0$

$$\frac{\partial\delta\rho}{\partial t} = -\rho_0\nabla\cdot\delta\underline{u}$$

$$\frac{\partial\delta\underline{u}}{\partial t} = -\frac{1}{\rho_0}\nabla\delta P \Rightarrow \frac{\partial^2\delta\underline{u}}{\partial t^2} = -\frac{1}{\rho_0}\nabla\frac{\partial\delta P}{\partial t}$$

$$\frac{\partial\delta P}{\partial t} = -\gamma P_0\nabla\cdot\delta\underline{u}$$

$$\frac{\partial^2 \delta \underline{u}}{\partial t^2} = + \frac{1}{\rho_0} \nabla \left(\gamma P_0 \nabla \cdot \delta \underset{\sim}{u} \right)$$

$$= \underbrace{\frac{\gamma P_0}{\rho_0}}_{c_s^2} \nabla \left(\nabla \cdot \delta \underset{\sim}{u} \right)$$

wave equation

c_s = sound speed