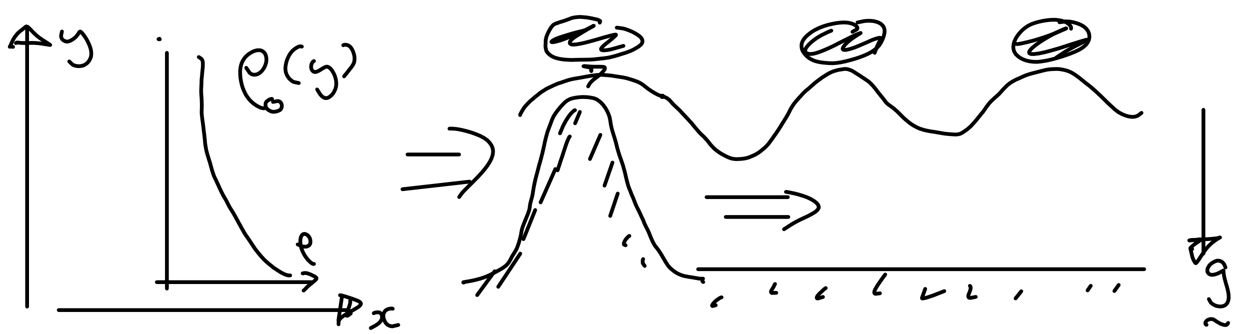
Buoyancy waves and Instabilities

Concepts

- Density stratification
- Waves and instabilities
- Dispersion relation



$$\rho = \rho_0 + \delta\rho$$

$$\rho\left[\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u}\right] = -\nabla p - \underset{\sim}{g}\rho$$

$$\underset{\sim}{u} = \cancel{\underset{\sim}{u}_0} + \delta\underset{\sim}{u}$$

static

$$p = P_0(y) + \delta P$$

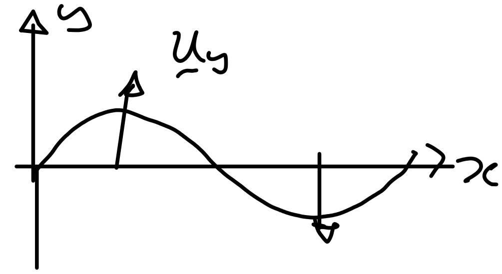
$$\rho_0\left[\frac{\partial \delta\underset{\sim}{u}}{\partial t}\right] = -\nabla\delta P - \underset{\sim}{g}\,\delta\rho$$

$$\rho_0\frac{\partial^2 \delta\underset{\sim}{u}}{\partial t^2} = \underbrace{-\nabla\frac{\partial \delta P}{\partial t}}_{\text{neglect for now}} - \underset{\sim}{g}\frac{\partial \delta\rho}{\partial t}$$

Density
$$\frac{\partial \rho}{\partial t} + \underset{\sim}{u}\cdot\nabla\rho = -\cancel{\rho\nabla\cdot\underset{\sim}{u}} \rightarrow 0 \quad \text{incompressible}$$

$$\frac{\partial \delta\rho}{\partial t} + \delta\underset{\sim}{u}\cdot\nabla\rho_0 = 0$$

$$\rho_0 \frac{\partial^2 \delta \underline{u}}{\partial t^2} = + \underline{g}\, \delta \underline{u} \cdot \nabla \rho_0$$



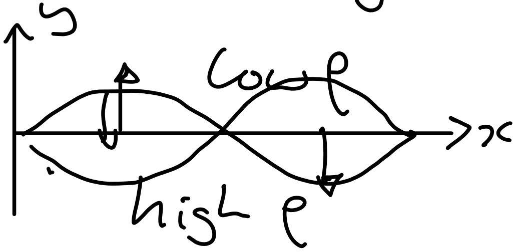
y component

$$\rho_0 \frac{\partial^2 \delta u_y}{\partial t^2} = g \frac{d\rho_0}{dy} \delta u_y$$

$$\frac{\partial^2 \delta u_y}{\partial t^2} = \underbrace{\frac{g}{\rho_0} \frac{\partial \rho_0}{\partial y}}_{-N^2} \delta u_y = -N^2 \delta u_y$$

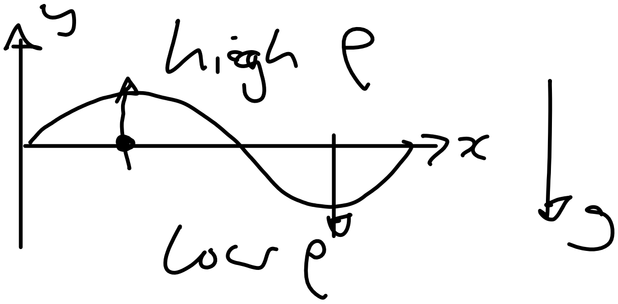
$$\delta u_y \propto e^{iNt}$$

$N > 0$ if $\frac{\partial \rho_0}{\partial y} < 0$



$$N = \sqrt{-\frac{g}{\rho_0} \frac{\partial \rho_0}{\partial y}}$$

N imaginary if $\frac{\partial \rho_0}{\partial y} > 0 \Rightarrow$ growing solutions



Rayleigh-Taylor instability

use $\nabla\cdot\underset{\sim}{u}=0$

in x-y $\quad \frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}=0$

$$\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial t}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u_y}{\partial t}\right)=0$$

$$\frac{\partial u_x}{\partial t}=-\frac{1}{\rho_0}\frac{\partial \delta p}{\partial x} \qquad \frac{1}{\rho_0}\frac{\partial^2\delta p}{\partial x^2}=\frac{\partial}{\partial y}\left(\frac{\partial u_y}{\partial t}\right)$$

$$\frac{\partial^2 u_y}{\partial t^2}=-\frac{1}{\rho_0}\frac{\partial}{\partial y}\left(\frac{\partial\delta p}{\partial t}\right)-\delta u_y N^2$$

$$\frac{\partial^2}{\partial x^2}\frac{\partial^2 u_y}{\partial t^2}=-\frac{1}{\rho_0}\frac{\partial}{\partial y}\left(\frac{\partial}{\partial t}\frac{\partial^2\delta p}{\partial x^2}\right)-\frac{\partial^2 u_y}{\partial x^2}N^2$$

$$\frac{\partial^2}{\partial x^2}\frac{\partial^2 u_y}{\partial t^2}=-\frac{1}{\rho_0}\frac{\partial}{\partial y}\frac{\partial}{\partial t}\left(\rho_0\frac{\partial}{\partial y}\frac{\partial u_y}{\partial t}\right)-N^2\frac{\partial^2 u_y}{\partial x^2}$$

$$\left(\frac{\partial^2}{\partial x^2}\frac{\partial^2 u_y}{\partial t^2}+\frac{1}{\rho_0}\frac{\partial}{\partial y}\left(\rho_0\frac{\partial}{\partial y}\frac{\partial^2 u_y}{\partial t^2}\right)\right)=-N^2\frac{\partial^2 u_y}{\partial x^2}$$

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 u_y}{\partial x^2}+\frac{1}{\rho_0}\frac{\partial}{\partial y}\left(\rho_0\frac{\partial u_y}{\partial y}\right)\right]=-N^2\frac{\partial^2 u_y}{\partial x^2}$$

$$u_y = a e^{i(kx + ly - \omega t)}$$

$$\frac{\partial^2}{\partial t^2} \to -\omega^2$$

$$\omega^2\left[k^2 + l^2\right] = N^2 k^2 \qquad \omega^2 = \frac{N^2k^2}{k^2+l^2}$$

phase speed in y

$$\frac{\omega}{l} = N\frac{k}{l}\frac{1}{\sqrt{k^2+l^2}} \qquad \frac{k}{l} > 0 \Rightarrow \text{up}$$

Group speed

$$\frac{\partial \omega}{\partial l} = \frac{Nk}{(k^2+l^2)^{3/2}}(-l) = \frac{-Nkl}{(k^2+l^2)} < 0 \text{ down}$$