Rayleigh-Taylor instability

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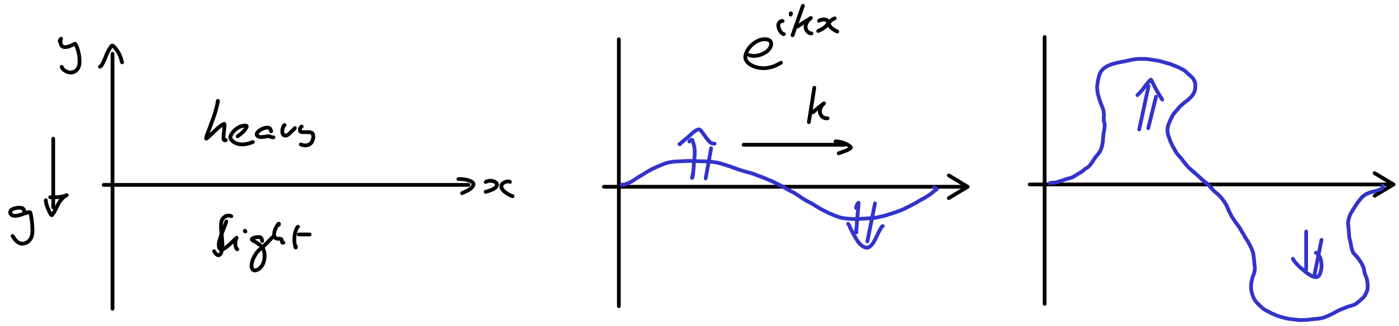
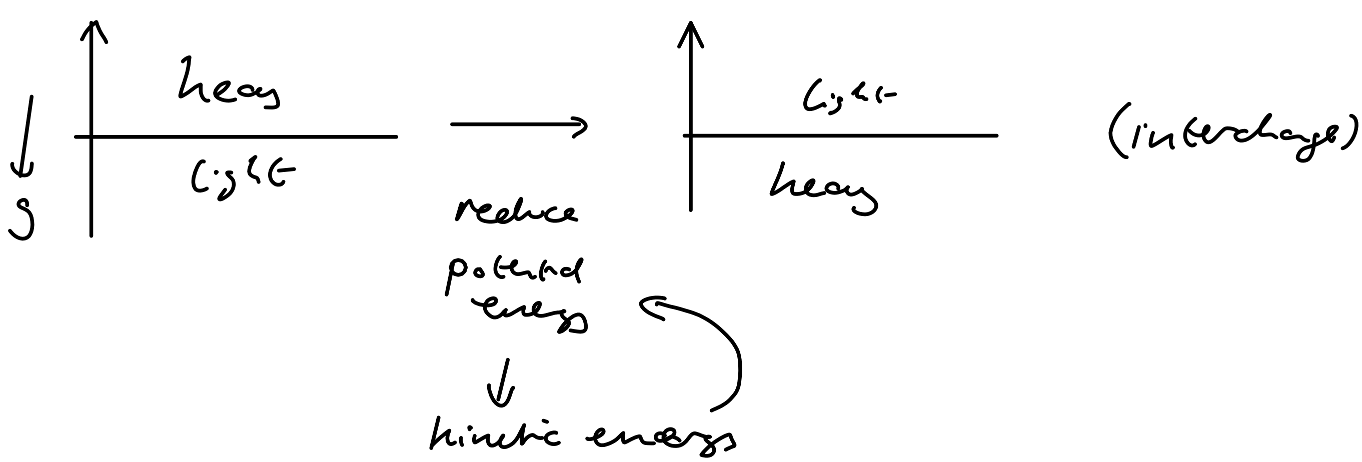
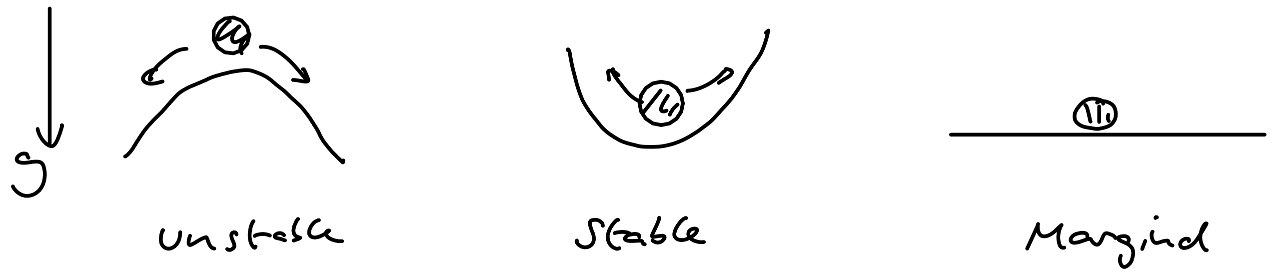


① Dynamical : Solve system of equations in time – look for growing solutions

② Energy principle

If a system can evolve to reduce its potential energy then it is unstable





Bouyancy waves

$$\omega = N\frac{k}{\sqrt{k^2+l^2}} \qquad e^{i(kx+ly-\omega t)}$$

$$N = \sqrt{-\frac{g}{\rho_0}\frac{\partial \rho}{\partial z}}$$

$\frac{\partial \rho}{\partial z} < 0$ stable

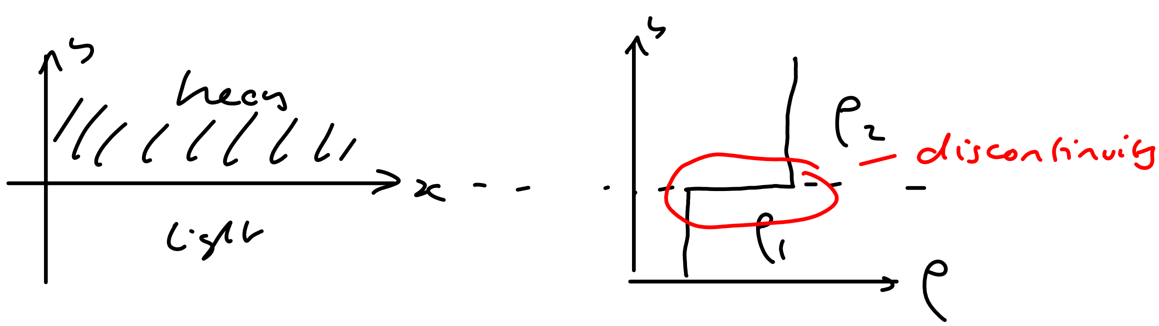
$\frac{\partial \rho}{\partial z} > 0$ unstable

ω imaginary

$\gamma = \mathrm{Im}(\omega)$

$\Rightarrow e^{\gamma t}$

Sharp boundary



Discontinuous solution - need to be careful with matching solutions on both sides

Equation of motion

$$\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 u_y}{\partial x^2} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\rho \frac{\partial u_y}{\partial y}\right)\right] = -N^2 \frac{\partial^2 u_y}{\partial x^2} \qquad *$$

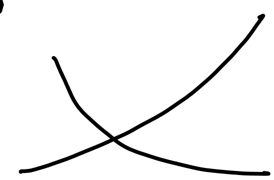
u_y - velocity in y direction.

This equation applies in each region $N = 0$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\left[\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right] = 0$$

$$u_y(t,x,y) = u_y(t,y)\, e^{ikx}$$

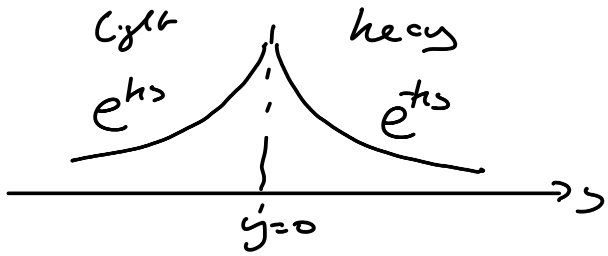
$$\Rightarrow \frac{\partial^2 u_y}{\partial y^2} - k^2 u_y = 0$$



$$u_y = A e^{\pm ky}$$

$u_y \to 0$ as $y \to \pm\infty$

$$\Rightarrow u_y(t,y) = \begin{cases} u_y(t)\, e^{-ky} & \text{if } y > 0 \\ u_y(t)\, e^{ky} & \text{if } y < 0 \end{cases}$$



next, integrate $*$ in y across narrow layer

δ

$$\lim_{\delta \to 0} \int_{-\delta/2}^{\delta/2}$$

$$\frac{\partial^2}{\partial t^2}\int \rho\left[\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}\right]dz = g\int \frac{\partial \rho}{\partial z}\frac{\partial^2 u_z}{\partial x^2}\,dz$$

$$\lim_{\delta\to 0}\frac{\partial^2}{\partial t^2}\left[\rho\left(\cancel{\frac{\partial^2 u_z}{\partial x^2}z} + \frac{\partial u_z}{\partial z}\right)\right]_{-\delta/2\ (\text{light})}^{\delta/2\ (\text{heavy})} = g\,\frac{\partial^2 u_z}{\partial x^2}\,\underbrace{\Delta\rho}_{\rho_2-\rho_1}$$

$-\omega^2$

u_z continuous

$-k^2$

$$\Delta\left(\rho\frac{\partial u_z}{\partial z}\right)$$

(difference from $+\delta/2$ to $-\delta/2$

$$[\rho_2(-k) - \rho_1 k]$$

$$-\omega^2(-k)(\rho_2+\rho_1) = g(-k^2)(\rho_2-\rho_1)$$

$$\omega^2 = -gk\boxed{\frac{\rho_2-\rho_1}{\rho_2+\rho_1}} < 0 \text{ if } \rho_2 > \rho_1$$

Atwood number

(unstable)

$$\gamma = \mathrm{Im}(\omega) = \sqrt{gkA}$$