Shocks

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Conservation laws

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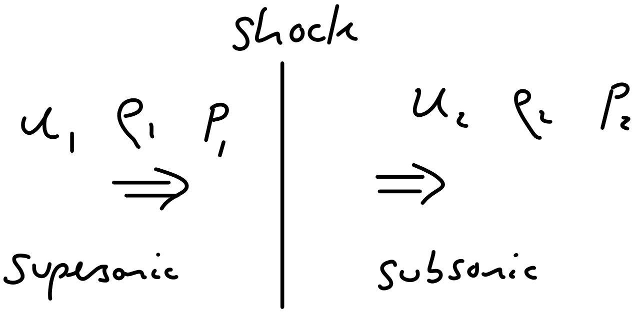

in frame of shock

Shock width ~ mean free path λ_{mfp}

e.g. Air at sea level $\lambda_{mfp} \simeq 70\text{nm}$

Mass

Mass flow into shock same as flow out

$$\rho_1 u_1 = \rho_2 u_2$$

$$\cancel{\frac{\partial}{\partial t}\rho} + \nabla\cdot(\rho \underline{u}) = 0$$

Conservation form

steady state

Integrate across shock

$$\lim_{\delta \to 0} \int_{-\delta/2}^{\delta/2}$$

$$\Rightarrow \quad \rho \underline{u} = \text{const}$$

Momentum

Write in conservative form

$$\cancel{\frac{\partial}{\partial t}(\rho \underline{u})} + \nabla\cdot\Big(\underbrace{p\,\underline{\underline{I}} + \rho\,\underline{u}\,\underline{u}}_{\text{Constant}}\Big) = 0$$

Consider components perpendicular to shock

$$\boxed{p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2}$$

Total pressure

static pressure

Dynamic pressure

"ram" pressure

Energy

energy density

$$\cancel{\frac{\partial E}{\partial t}} + \nabla\cdot\Big[\underbrace{(E+p)\,\underline{u}}_{\text{Const}}\Big] = 0$$

$$E = \frac{3}{2}p + \frac{1}{2}\rho u^2$$

internal energy

kinetic energy

$$\Rightarrow \left[\frac{5}{2}\frac{p}{\rho} + \frac{1}{2}u^2\right]\underbrace{\rho u}_{\text{Const}} = \text{const}$$

$$\Rightarrow \frac{5}{2}\frac{p}{\rho} + \frac{1}{2}u^2 = \text{const}$$

Bernoulli's equation

$$\frac{5}{2}\frac{p_1}{\rho_1} + \frac{1}{2}u_1^2 = \frac{5}{2}\frac{p_2}{\rho_2} + \frac{1}{2}u_2^2$$

Three equations:

Three unknowns

$\rho_2 , u_2 \ P_2$

$$\boxed{\begin{gathered} \rho_1 u_1 = \rho_2 u_2 \\ P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2 \\ \tfrac{1}{2} u_1^2 + \frac{\gamma}{\gamma-1} \frac{P_1}{\rho_1} = \tfrac{1}{2} u_2^2 + \frac{\gamma}{\gamma-1} \frac{P_2}{\rho_2} \end{gathered}}$$

$\Rightarrow$ Give a solution for ρ_2, u_2, P_2 given upstream $\rho_1 \ u_1 \ P_1$

① Limit to compression

$$\frac{u_1}{u_2} = \frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_1^2}{(\gamma-1) M_1^2 + 2}$$

M_1 Mach number $= u_1 / c_{s1}$

as $M_1 \to \infty$ $\quad \dfrac{\rho_2}{\rho_1} \to \dfrac{\gamma+1}{\gamma-1}$ $\quad \gamma = 5/3$ $\to 4$

Maximum density compression in a single shock

$\Rightarrow$ need multiple shocks for high compression (e.g. ICF)

② Entropy constraint

entropy $S = c_v \ln (P / \rho^\gamma) + \text{const}$

change in S, ΔS across shock

$$\Delta S = c_v \left[\ln (P_2 / \rho_2^\gamma) - \ln (P_1 / \rho_1^\gamma) \right]$$

Must have $\Delta S \geq 0$ since $\Delta S < 0$ forbidden by 2nd law of Thermodynamics

$$\frac{d\Delta S}{dM_1} = 0 \quad \text{at } M_1 = 1$$

$$\frac{d^2\Delta S}{dM_1^2} = 0 \qquad \frac{d^3\Delta S}{dM_1^3} > 0 \quad \text{at } M_1 = 1$$

$\Rightarrow$ $M_1 > 1$ $\quad \Delta S > 0$

$M_1 < 1$ $\quad \Delta S < 0$ ✗

$\Rightarrow$ Mach number into shock must be > 1

flow slows down $\Rightarrow$ compression