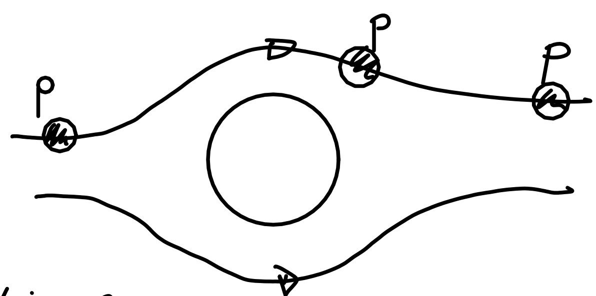
Incompressible flow

Concepts

- Convective derivative
- Mach number
- Characteristic scales and normalisation
- Singular limits

When is flow incompressible?



Euler equations

pressure
$$\frac{\partial p}{\partial t} + \underset{\sim}{u} \cdot \nabla p = -\gamma p \underbrace{\nabla \cdot \underset{\sim}{u}}_{=0} \quad \text{incompressible}$$

p constant along flow

ρ " " "

Momentum

$$\cancel{\frac{\partial \underset{\sim}{u}}{\partial t}} + \underset{\sim}{u} \cdot \nabla \underline{u} = -\frac{1}{\rho} \nabla p$$

Steady state

$$u_x \frac{du_x}{dx} = -\frac{1}{\rho} \frac{dP}{dx}$$

$$u_x^2 \frac{du_x}{u_x} = -\frac{P}{\rho} \frac{dP}{P}$$

$$\underbrace{\gamma \frac{u_x^2}{c_s^2}}_{M^2} \frac{du_x}{u_x} = -\frac{dP}{P} \qquad \frac{1}{\gamma}c_s^2$$

M^2 $\quad M$ = Mach number = $\frac{u_x}{c_s}$

$\Rightarrow$ if M small (< 1)

$\Rightarrow$ $\frac{dP}{P}$ relative pressure changes small

Normalisation and Singular limits

Normalise : Turn equations into dimensionless form

$$\rho = \rho_0 \hat{\rho} \qquad \underset{\sim}{u} = u_0 \underset{\sim}{\hat{u}} \qquad \nabla = \frac{1}{L}\hat{\nabla}$$

$\hat{\rho}$ ↖ $O(1)$

Density equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underset{\sim}{u}) = 0 \qquad \cancel{\rho_0}\frac{\partial \hat{\rho}}{\partial t} + \frac{\hat{\nabla}}{L}\cdot(\hat{\rho}\hat{u})\cancel{\rho_0} u_0 = 0$$

$$t = \frac{L}{u_0}\hat{t} \qquad \frac{\partial \hat{\rho}}{\partial \hat{t}} + \hat{\nabla}\cdot(\hat{\rho}\hat{u}) = 0$$

Momentum

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u} = -\frac{1}{\rho}\nabla P \qquad P = P_0\hat{P}$$

$$\frac{u_0}{L}u_0\left[\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u}\cdot\nabla\hat{u}\right] = \frac{1}{\rho_0\hat{\rho}}\frac{P_0}{L}\nabla\hat{P}$$

$$\overbrace{\qquad\qquad}^{}$$
$$\frac{1}{L}\;\frac{1}{\gamma}c_s^2$$

$$\gamma \underbrace{\frac{u_0^2}{c_s^2}}_{M^2} \left[\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u}\cdot\nabla\hat{u} \right] = \frac{-1}{\hat{\rho}} \nabla \hat{p}$$

$\nabla p \rightarrow 0$ as $M \rightarrow 0$

$$\hat{p} = \bar{p} + M \hat{p}_1$$

$$\nabla \hat{p} = M \nabla \hat{p}_1$$

$$\frac{\partial \hat{p}}{\partial \hat{t}} = M \frac{\partial \hat{p}_1}{\partial t}$$

<u>Pressure equation</u>

$$\frac{\partial p}{\partial t} + \underset{\sim}{u}\cdot\nabla p = -\gamma p \nabla\cdot\underline{u}$$

$$\cancel{p_0} M \cancel{\frac{u_0}{L}} \left[\frac{\partial \hat{p}_1}{\partial \hat{t}} + \hat{u}\cdot\nabla\hat{p}_1 \right] = -\gamma \cancel{p_0} \hat{p} \cancel{\frac{u_0}{L}} \hat{\nabla}\cdot\hat{u}$$

$$M \left[\frac{\partial \hat{p}_1}{\partial \hat{t}} + \hat{u}\cdot\nabla\hat{p}_1 \right] = \underbrace{-\gamma \hat{p} \nabla\cdot\hat{u}}$$

$\rightarrow 0$ as $M \rightarrow 0$

$\Rightarrow \nabla\cdot\underset{\sim}{u} \rightarrow 0$