Incompressible flow II

Concepts

- Normalisation
- Singular limits

More general proof that $\nabla \cdot u = 0$ when Mach number small

Normalisation → removing dimensions

typical scales

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{u}) = 0$$

$$\rho = \rho_0 \hat{\rho} \leftarrow O(1)$$

$$\underset{\sim}{u} = u_0 \hat{\underset{\sim}{u}}$$

$$\nabla = \frac{1}{L} \hat{\nabla}$$

$$\cancel{\rho_0} \frac{\partial \hat{\rho}}{\partial t} + \hat{\nabla} \cdot (\hat{\rho} \hat{\underset{\sim}{u}}) \frac{1}{L} \cancel{\rho_0} u_0 = 0$$

$$t = \frac{L}{u_0} \hat{t} \qquad \frac{\partial \hat{\rho}}{\partial \hat{t}} + \hat{\nabla} \cdot (\hat{\rho} \hat{\underset{\sim}{u}}) = 0$$

Momentum

$$\frac{\partial u}{\partial \hat{t}} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} = -\frac{1}{\rho} \nabla P \qquad P = P_0 \hat{P}$$

$$\frac{u_0 u_0}{\cancel{L}} \left[\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u} \cdot \hat{\nabla} \hat{u} \right] = \frac{-1}{\rho_0 \hat{\rho}} \frac{1}{\cancel{L}} P_0 \hat{\nabla} \hat{P}$$

$$\frac{P_0}{\rho_0} = c_s^2 \frac{1}{\gamma}$$

$$\underbrace{\gamma \frac{u_0^2}{c_s^2}}_{M^2} \left[\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u} \cdot \nabla \hat{u} \right] = - \frac{1}{\hat{\rho}} \hat{\nabla} \hat{p}$$

As $M \to 0 \quad \hat{\nabla} \hat{p} \to 0$

$$\hat{p} = \bar{p} + M \hat{p}_1 \quad \left(+ M^2 \hat{p}_2 + M^3 \hat{p}_3 + \dots \right)$$

Pressure

$$\frac{\partial p}{\partial t} + \underline{u} \cdot \nabla p = - \gamma p \nabla \cdot \underline{u}$$

$$\frac{\partial \hat{p}}{\partial t} = M \frac{\partial \hat{p}_1}{\partial t}$$

$$\cancel{\frac{u_0}{L}} \cancel{p_0} M \left[\frac{\partial \hat{p}_1}{\partial \hat{t}} + \hat{u} \cdot \hat{\nabla} \hat{p}_1 \right] = - \gamma \cancel{p_0} \hat{p} \cancel{\frac{u_0}{L}} \hat{\nabla} \cdot \hat{u}$$

$$\nabla \hat{p} = M \nabla \hat{p}_1$$

$$M \left[\frac{\partial \hat{p}_1}{\partial \hat{t}} + \hat{u} \cdot \hat{\nabla} \hat{p}_1 \right] = - \gamma \hat{p} \hat{\nabla} \cdot \hat{u}$$

as $M \to 0 \quad \nabla \cdot \underline{u} \to 0$

Singular limits