Von Neumann's method

Concepts

- Stability and instability
- Von Neumann's method

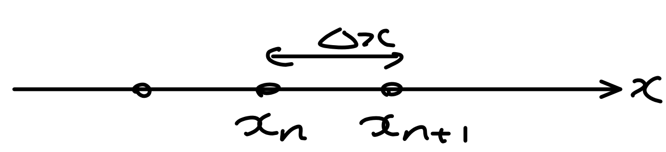
A numerical method is stable if there are no unphysical growing modes (waves)

Methods can be:

- Unconditionally unstable
- Unconditionally stable
- Conditionally stable

Von Neumann's Method

$$\frac{\partial \rho}{\partial t} = -u_x \frac{\partial \rho}{\partial x} \qquad \left.\frac{\partial \rho}{\partial x}\right|_{x_n} \simeq \frac{\rho(x_{n+1}) - \rho(x_n)}{\Delta x}$$



$$\rho(t,x) = A(t)\, e^{ikx}$$

$$\rho(x_{n+1}) = A(t)\, e^{ikx_{n+1}} \qquad x_{n+1} = x_n + \Delta x$$

$$= A(t)\, e^{ikx_n}\, e^{ik\Delta x}$$

$$\frac{\partial A}{\partial t} \cancel{e^{ikx_n}} = -u_x \left[\frac{A \cancel{e^{ikx_n}} e^{ik\Delta x} - A \cancel{e^{ikx_n}}}{\Delta x} \right]$$

$$\frac{\partial A}{\partial t} = \underbrace{-\frac{u_x}{\Delta x}\left(e^{ik\Delta x} - 1\right)}_{\alpha(k)} A$$

$$\frac{\partial A}{\partial t} = -\alpha A \qquad A(t) = A_0 e^{-\alpha t}$$

if $\alpha > 0 \Rightarrow A(t)$ shrinks

$\longrightarrow 0$ as $t \to \infty$

$\alpha < 0 \Rightarrow A(t)$ grows

unstable

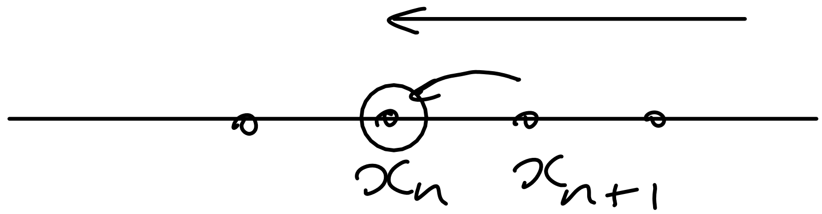
$$\mathrm{Re}(\alpha) = \frac{u_x}{\Delta x}\left(\underbrace{\cos(k\Delta x)}_{1 \to -1} - 1\right)$$

$$-2 < \cos(k\Delta x) - 1 < 0 \Rightarrow u_x < 0$$

for stability

Upwinding

$u_x < 0$ Stable



Downwindins

$u_x > 0$ Unstable