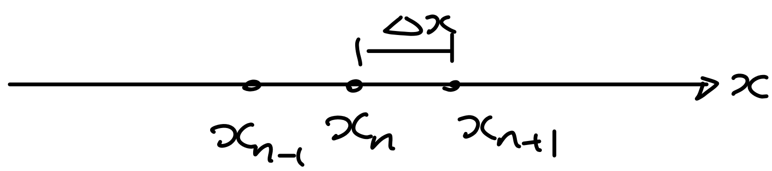
The CFL condition

Concepts

- Time integration
- Courant number
- Courant-Friedrichs-Lewy condition

$$\frac{\partial \rho}{\partial t} = -u_x \frac{\partial \rho}{\partial x}$$



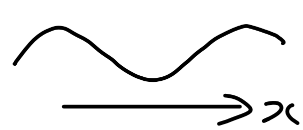
$$\left.\frac{\partial \rho}{\partial x}\right|_{x_n} \simeq \frac{\rho(x_{n+1}) - \rho(x_n)}{\Delta x} + O(\Delta x)$$

$$\frac{\partial \rho}{\partial t} \simeq \frac{\rho(t_{j+1}) - \rho(t_j)}{\Delta t} + O(\Delta t)$$

$$\underbrace{\frac{\rho(t_{j+1}) - \rho(t_j)}{\Delta t}}_{\text{at } x_n} = -u_x \underbrace{\frac{\rho(x_{n+1}) - \rho(x_n)}{\Delta x}}_{\text{at } t_j}$$

Von Neumann's method

$$\rho(t,x) = A(t)\, e^{ikx}$$



$$\frac{A(t_{j+1}) - A(t_j)}{\Delta t} e^{ik x_n} = -u_x A(t_j) \frac{e^{ik x_{n+1}} - e^{ik x_n}}{\Delta x}$$

$$x_{n+1} = x_n + \Delta x$$

$$e^{ik x_{n+1}} = e^{ik x_n} e^{ik \Delta x}$$

$$\frac{A(t_{j+1}) - A(t_j)}{A(t_j)} = -\frac{u_x \Delta t}{\Delta x} \left(e^{ik\Delta x} - 1 \right)$$

$$\frac{A(t_{j+1})}{A(t_j)} = 1 - \underbrace{\frac{u_x \Delta t}{\Delta x}}_{\mu \text{ courant number}} \left(e^{ik\Delta x} - 1 \right) \underset{\text{for stability}}{\leqslant 1}$$

A(t) can be complex

for stability, amplitude is important

$$\left| \frac{A(t_{j+1})}{A(t_j)} \right|^2 = \left(1 - \mu\left(e^{ik\Delta x} - 1\right)\right)\left(1 - \mu\left(e^{-ik\Delta x} - 1\right)\right)$$

$$= 1 - \mu\left(e^{ik\Delta x} + e^{-ik\Delta x} - 2\right)$$

$$+ \mu^2 \underbrace{\left(e^{ik\Delta x} - 1\right)\left(e^{-ik\Delta x} - 1\right)}_{2 - e^{ik\Delta x} - e^{-ik\Delta x}}$$

$$e^{ik\Delta x} + e^{-ik\Delta x} = 2\cos(k\Delta x)$$

$$\left|\frac{A(t_{j+1})}{A(t_j)}\right|^2 = 1 - (\mu+\mu^2)\,2\underbrace{(\cos(k\Delta x) - 1)}_{\leqslant 0}$$

$\Rightarrow \mu + \mu^2 \leqslant 0$ for stability

where is this unstable?

1. $\mu > 0 \Rightarrow$ unstable

   $\mu = \dfrac{u_x \Delta t}{\Delta x} \Rightarrow u_x < 0$ for stability

2. if $\mu < -1$ $\quad \left|\dfrac{u_x \Delta t}{\Delta x}\right| < 1$

$\Rightarrow$ $-1 < \mu < 0$ stable

$$\boxed{\Delta t < \left|\frac{\Delta x}{u_x}\right|}$$ CFL condition