Grad-Shafranov equation

Contents

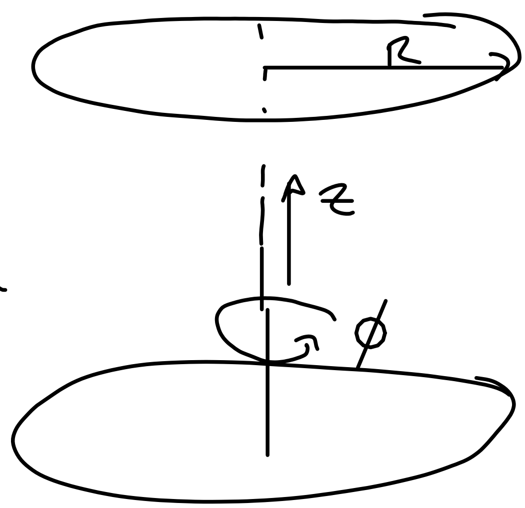
- Force balance in an axisymmetric torus
- Poloidal and toroidal magnetic fields
- Poloidal flux
- Flux surfaces

$$\nabla \cdot \underset{\sim}{B} = 0 \qquad \nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J} \qquad \underset{\sim}{J} \times \underset{\sim}{B} = \nabla P$$

<u>$\nabla \cdot \underset{\sim}{B} = 0$</u>

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$$

↖ vector potential



R - Major radius

ϕ toroidal angle

axisymmic
constant in ϕ

$$\frac{\partial}{\partial \phi} \to 0$$

$$\underset{\sim}{B} = \nabla \times \underset{\sim}{A} = \left[\cancel{\frac{1}{R}\frac{\partial A_z}{\partial \phi}} - \frac{\partial A_\phi}{\partial z} \right] \underset{\sim}{\hat{R}}$$

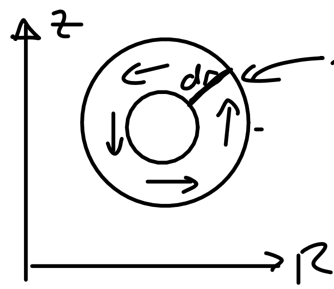
$$+ \underbrace{\left[\frac{\partial A_R}{\partial z} - \frac{\partial A_z}{\partial R} \right]}_{B_\phi \text{ toroidal field}} \underset{\sim}{\hat{\phi}}$$

$$+\left[\frac{1}{R}\frac{\partial}{\partial R}(RA_\phi) - \cancel{\frac{\partial A_R}{\partial \phi}}\right]\hat{z}$$

$$\underset{\sim}{B} = \underbrace{-\frac{\partial A_\phi}{\partial z}\hat{R} + \frac{1}{R}\frac{\partial}{\partial R}(RA_\phi)\hat{z}}_{\text{poloidal field } \underset{\sim}{B}_p} + \underbrace{B_\phi \hat{\phi}}_{\text{toroidal field}}$$

$$= -\frac{1}{R}\frac{\partial(A_\phi R)}{\partial z}\hat{R} + \frac{1}{R}\frac{\partial}{\partial R}(RA_\phi)\hat{z} + B_\phi\hat{\phi}$$

$A_\phi R = \psi$ poloidal flux



ψ = constant

$\underline{B}_p$ along surfaces of const ψ

$$d\psi \sim B_p \underbrace{R\,dr}_{\text{area}}$$

$$\underset{\sim}{B} = \underbrace{\frac{1}{R}\nabla\psi \times \hat{\phi}}_{\underline{B}_p} + \underbrace{B_\phi\hat{\phi}}_{\text{toroidal}}$$

Ampere's law

$$\nabla \times \underline{B} = \mu_0 \underline{J}$$

$$\nabla\times\underline{B} = \underbrace{\nabla\times\left(\frac{1}{R}\nabla\psi\times\hat{\phi}\right)} + \underbrace{\nabla\times\left(B_\phi\hat{\phi}\right)}$$

$$\cancel{\frac{1}{R}\nabla\psi(\nabla\cdot\hat{\phi})}^{\,0} \qquad \nabla\times\left(RB_\phi\frac{\hat{\phi}}{R}\right)$$

$$-\hat{\phi}\,\nabla\cdot\left(\frac{1}{R}\nabla\psi\right) \qquad = RB_\phi\,\cancel{\nabla\times\left(\frac{\hat{\phi}}{R}\right)}^{\,0}$$

$$+\underbrace{\cancel{\hat{\phi}\cdot\nabla}}_{\partial/\partial\phi}\left(\frac{1}{R}\nabla\psi\right) \qquad + \nabla(RB_\phi)\times\frac{\hat{\phi}}{R}$$

$$-\cancel{\frac{1}{R}\nabla\psi\cdot\nabla\hat{\phi}}^{\,0}$$

$$\nabla\times\underline{B} = \underbrace{-\hat{\phi}\,\nabla\cdot\left(\frac{1}{R}\nabla\psi\right)}_{\mu_0 J_\phi\hat{\phi}} + \nabla(RB_\phi)\times\frac{\hat{\phi}}{R} = \mu_0\underline{J}$$

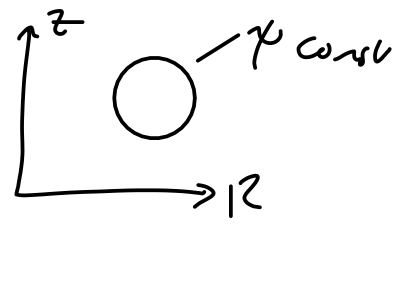
poloidal current

$\underline{J}\times\underline{B} = \nabla P$

$$\underline{B}\cdot\nabla P = 0 \qquad \underline{B}_p\cdot\nabla P = 0$$

because $\hat{\phi}\cdot\nabla P = 0$

$$\left(\frac{1}{R}\nabla\psi\times\hat{\phi}\right)\cdot\nabla P = 0$$



$$(\nabla\psi\times\nabla P)\cdot\hat{\phi} = 0$$

$$\nabla P \propto \nabla\psi \qquad P = P(\psi)$$

$$= \frac{\partial P}{\partial\psi}\nabla\psi$$

$$\underset{\sim}{J} \cdot \nabla P = 0$$

$$\left(\nabla(RB_\phi) \times \frac{\hat{\phi}}{R} \right) \cdot \nabla P = 0$$

$$\left(\nabla(RB_\phi) \times \nabla P \right) \cdot \hat{\phi} = 0$$

$$\nabla(\underbrace{RB_\phi}_{f(\psi)}) \propto \nabla \psi$$

$$\underset{\sim}{J} \times \underline{B} = \left(J_\phi \hat{\phi} + \frac{1}{\mu_0 R} \nabla f \times \hat{\phi} \right) \times \left(\frac{1}{R} \nabla \psi \times \hat{\phi} + B_\phi \hat{\phi} \right)$$

$$= \underbrace{J_\phi \hat{\phi} \times \frac{1}{R} (\nabla \psi \times \hat{\phi})}_{\frac{1}{R}\left(\nabla\psi(\hat{\phi}\cdot\hat{\phi}) - \hat{\phi}\,\cancel{(\hat{\phi}\cdot\nabla\psi)}\right)} + \frac{1}{\mu_0 R} (\nabla f \times \hat{\phi}) \times B_\phi \hat{\phi}$$

$$f(\psi)$$

$$\frac{\partial f}{\partial \psi} \cdot \nabla \psi$$

$$\underset{\sim}{J} \times \underline{B} = J_\phi \frac{1}{R} \nabla \psi - \underbrace{\frac{1}{\mu_0 R} B_\phi}_{\frac{RB_\phi}{\mu_0 R^2} = \frac{f}{\mu_0 R^2}} \frac{\partial f}{\partial \psi} \nabla \psi = \nabla P = \frac{\partial P}{\partial \psi} \nabla \psi$$

because
$p = p(\psi)$

$$\frac{\partial P}{\partial \psi} = \frac{1}{\mu_0 R^2} \left[\mu_0 R J_\phi - f \frac{\partial f}{\partial \psi} \right]$$

$$\frac{1}{\mu_0} \nabla \cdot \left(\frac{1}{R} \nabla \psi \right)$$

$$\boxed{R \nabla \cdot \left(\frac{1}{R} \nabla \psi \right) = -\mu_0 R^2 \frac{\partial p}{\partial \psi} - f \frac{\partial f}{\partial \psi}}$$

Grad - Shafranov

$$\frac{\partial p}{\partial \psi} = p' \qquad \frac{\partial f}{\partial \psi} = f'$$

$$\underline{\Delta^* \psi = -\mu_0 R^2 p' - f f'}$$

$$\psi(R, z) \qquad p(\psi), \; f(\psi)$$