Large aspect ratio

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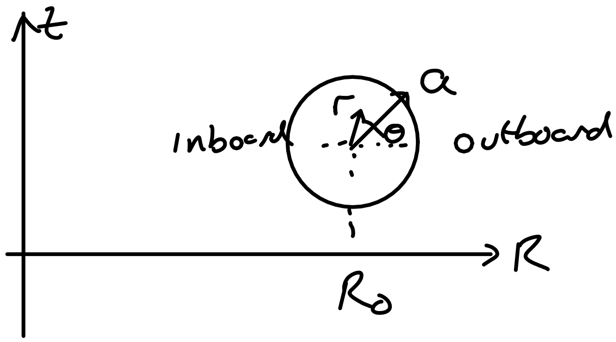
  of Grad-shafranov equation

Grad Shafranov

$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi}{\partial R}\right) + \frac{\partial^2 \psi}{\partial z^2} = -\mu_0 R^2 \frac{\partial p}{\partial \psi} - f\frac{\partial f}{\partial \psi}$$

$\psi(R,z) \qquad p(\psi) \qquad f(\psi) = RB_\phi$



$\frac{R_0}{a}$ aspect ratio

$\varepsilon = a/R_0 \ll 1$

toroidal coordinates (r, θ, ϕ)

ϕ ← toroidal angle

θ ← poloidal angle

$$R = R_0 + r\cos\theta$$
$$z = r\sin\theta$$

$$\underline{\underline{\frac{\partial^2 \psi}{\partial R^2}}} - \frac{1}{R}\frac{\partial \psi}{\partial R} + \underline{\underline{\frac{\partial^2 \psi}{\partial z^2}}} = \mu_0 R^2 \overset{\partial p/\partial \psi}{p'} - ff'$$

Laplacian in 2D

Polar coordinates r, θ

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$$

$$r^2 = (R-R_0)^2 + z^2$$

$$\frac{\partial\psi}{\partial R} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial R}$$

$$\frac{\partial r}{\partial R} = \cos\theta$$

$$\tan\theta = \frac{z}{R-R_0}$$

$$\frac{\partial\theta}{\partial R} = -\frac{\sin\theta}{r}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi - \frac{1}{R}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\psi$$

$$= -\mu_0 R^2 p' - ff'$$

Large aspect ratio

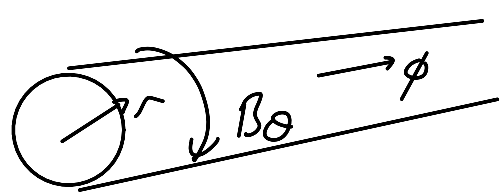
$$\psi(r,\theta) \simeq \underline{\psi_0(r)} + \psi_1(r,\theta)$$

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\psi_0}{\partial r} - \underbrace{\frac{1}{R}\cos\theta\frac{\partial\psi_0}{\partial r}}_{\text{small}} = -\mu_0 p' - ff'$$

cylindrical solution

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\psi_0}{\partial r} = -\mu_0 R^2 p' - ff'}$$

$$B_\theta = \frac{1}{R_0}\frac{\partial\psi_0}{\partial r}$$



$$f = R_0 B_\phi$$

$$\frac{R_0}{r}B_\theta + R_0\frac{\partial B_\theta}{\partial r} = -\mu_0 R^2 \underbrace{\frac{\partial p}{\partial\psi}}_{\frac{\partial p}{\partial r}\frac{dr}{d\psi}} - f\frac{\partial f}{\partial r}\underbrace{\frac{dr}{d\psi}}_{\frac{1}{B_\theta R_0}}$$

$$\frac{B_\theta^2}{r} + \underbrace{B_\theta \frac{\partial B_\theta}{\partial r}}_{\frac{\partial}{\partial r}\left(\frac{B_\theta^2}{2}\right)} = -\mu_0 \frac{dP}{dr} - \underbrace{B_\phi \frac{dB_\phi}{dr}}_{\frac{\partial}{\partial r}\left(\frac{B_\phi^2}{2}\right)}$$

$$\boxed{\frac{d}{dr}\left(\frac{B_\theta^2}{2\mu_0} + \frac{B_\phi^2}{2\mu_0} + P\right) + \frac{B_\theta^2}{\mu_0 r} = 0}$$

force balance