Lecture 10

Moving wings and real aircraft

So far we have considered flow around a stationary wing, and seen that there must be circulation Γ in order to keep the flow smooth.

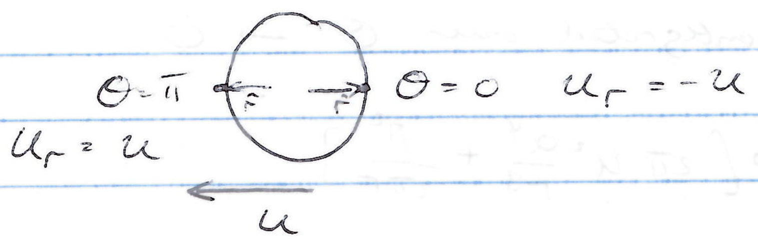
For flow at speed u past a stationary wing

$$\phi = u\left(r + \frac{a^2}{r}\right)\cos\theta + \frac{\Gamma\theta}{2\pi}$$

For a wing moving at speed u through a stationary fluid

$$\phi = u\frac{a^2}{r}\cos\theta + \frac{\Gamma\theta}{2\pi}$$

check $\quad u_r = \dfrac{\partial\phi}{\partial r} = -u\dfrac{a^2}{r^2}\cos\theta$

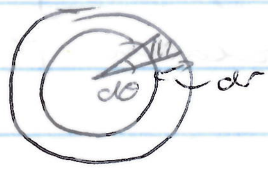


$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -u\frac{a^2}{r^2}\sin\theta + \frac{\Gamma}{2\pi r}$$

What is the kinetic energy of the fluid?

$$K.E. = \int \tfrac{1}{2}\rho|u|^2\, dV = \int \tfrac{1}{2}\rho\,(u_r^2 + u_\theta^2)\, dV$$

Volume element $\quad dV = r\, dr\, d\theta$



Integrating over all θ, and from $r = a$ to $r \to \infty$

$$K.E. = \int_a^\infty dr \int_0^{2\pi} d\theta\ \tfrac{1}{2}\rho\left[\left(-u\frac{a^2}{r^2}\cos\theta\right)^2 + \left(-\frac{ua^2}{r^2}\sin\theta + \frac{\Gamma}{2\pi r}\right)^2\right] r$$

note: $\cos^2\theta + \sin^2\theta = 1$

$$= \int_a^\infty dr \int_0^{2\pi} d\theta\ \tfrac{1}{2}\rho\left[u^2\frac{a^4}{r^3} - 2u\frac{a^2}{r}\sin\theta\frac{\Gamma}{2\pi r} + \frac{\Gamma^2}{4\pi^2 r}\right]$$

$\sin\theta$ integrated over $\theta \to 0$

$$K.E. = \int_a^\infty dr\ \tfrac{1}{2}\rho\left[2\pi u^2\frac{a^4}{r^3} + \frac{\Gamma^2}{2\pi r}\right]$$

$$= \tfrac{1}{2}\rho\left[\underbrace{-\pi u^2\frac{a^4}{r^2}}_{\tfrac{1}{2}\underbrace{\rho\pi a^2}_{M^*} u^2} + \underbrace{\frac{\Gamma^2}{2\pi}\ln r}_{\to\,\infty\,!}\right]_a^\infty$$

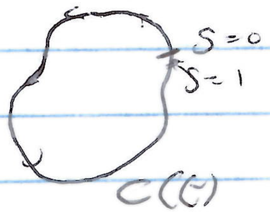
Depends on boundary conditions.

effective Mass of fluid dragged along by moving body.

10.2

Kelvin's circulation theorem

$$\frac{d}{dt}\int_{C(t)} \underset{\sim}{u}\cdot dx = \int_{C(t)} \frac{Du}{Dt}\cdot dx$$



$$\underset{\sim}{x} = \underset{\sim}{x}(s,t)$$

$$dx = \frac{\partial x}{\partial s}\, ds$$

$$\frac{d}{dt}\int_{C(t)} \underset{\sim}{u}\cdot d\underset{\sim}{x} = \cancel{\frac{d}{dt}}\int_0^1 \frac{\partial}{\partial t}\left(\underset{\sim}{u}\cdot\frac{\partial \underset{\sim}{x}}{\partial s}\right) ds$$

fixed s

$$\int_0^1 \frac{\partial \underset{\sim}{u}}{\partial t}\cdot\frac{\partial x}{\partial s}\, ds + \int_0^1 \underset{\sim}{u}\cdot\frac{\partial}{\partial s}\underbrace{\frac{\partial \underset{\sim}{x}}{\partial t}}_{\underset{\sim}{u}}\, ds$$

$$\frac{d}{dt}\oint_{C(t)} \underset{\sim}{u}\cdot d\underset{\sim}{x} = \oint \frac{dv}{dt}\cdot dx + \oint \underset{\sim}{v}\cdot \underbrace{\frac{d(d\underset{\sim}{x})}{dt}}_{dv}$$

change of v following the fluid

$$\frac{dv}{dt} = -\nabla\left(\frac{p}{\rho} + \chi\right)$$

$$\Rightarrow \quad \frac{d}{dt}\oint_{C(t)} \underset{\sim}{u}\cdot dx = -\oint \nabla\left(\frac{p}{\rho} - \frac{1}{2}u^2 + \chi\right) dx$$

$$= -\left[\frac{p}{\rho} - \frac{1}{2}u^2 + \chi\right]_C$$

$$= 0$$

- Does not need to be incompressible.
- Does not need to be 'simply connected'

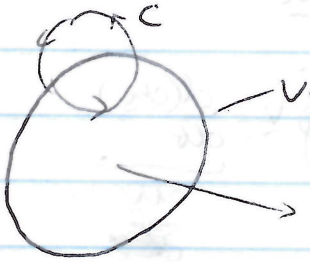
using Stokes' theorem

$$\oint_C u \cdot dx = \int_S \underset{\sim}{\omega} \cdot \underset{\sim}{n} \, dS$$

Inviscid, incompressible flow
If $\underset{\sim}{\omega} = 0$ initially,
then it is always zero

→ Cauchy Lagrange theorem

Shown easily in 2D, in 3D use Kelvin's circulation



vortex moves with fluid

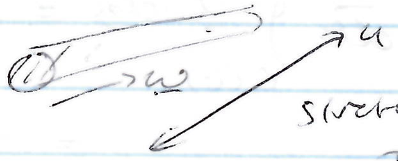
→ persistence of vortex rings

Vortex stretching

$$\frac{D\omega}{dt} = (\underset{\sim}{\omega} \cdot \nabla) u$$

e.g. 2D
$\omega = (0, 0, \omega_z)$

$$\frac{\partial \omega}{\partial t} = \omega_z \frac{\partial u}{\partial z}$$



stretching
$\rightarrow \frac{D\omega}{Dt} > 0$

$$\frac{d}{dt} \int_S \underset{\sim}{\omega} \cdot \underset{\sim}{n} \, ds = 0$$

⇒ shrinking

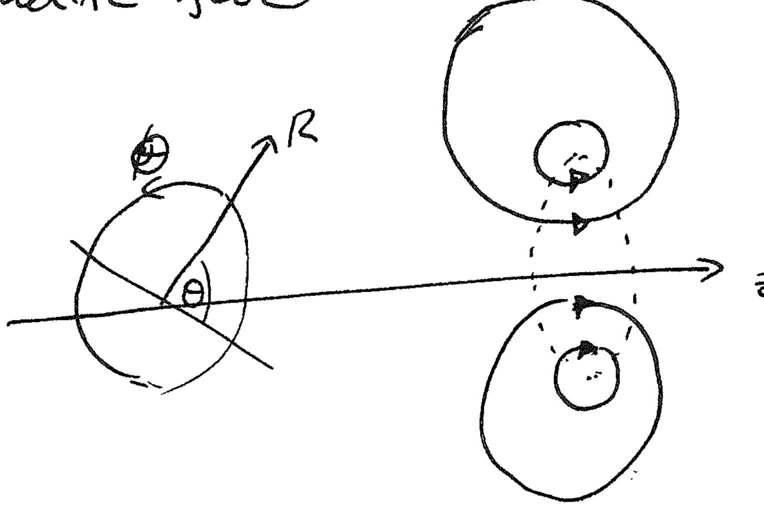
increase ω   e.g. Tornadoes

10.3

Vorticity equation in cylindrical coordinates

$$\frac{D\underset{\sim}{\omega}}{Dt} = (\underset{\sim}{\omega}\cdot\nabla)\underline{u}$$

axisymmetric flow



Vortex ring

If flow is constant in θ e.g. vortex ring

then $\underline{u} = u_R(R,z,t)\,\hat{\underline{e}}_R + u_z(R,z,t)\,\hat{\underline{e}}_z$

~~By the~~ Kelvin's circulation theorem
→ vorticity moves with flow so that
the vortex ring keeps its vorticity as it moves

$\underset{\sim}{\omega} = \nabla\times\underline{u}$ only non-zero component is

$$\underset{\sim}{\omega} = \omega_\theta\,\hat{\underline{e}}_\theta = \left(\frac{\partial u_R}{\partial z} - \frac{\partial u_z}{\partial R}\right)\hat{e}_\theta$$

$$\frac{D\underset{\sim}{\omega}}{Dt} = \frac{\partial\underset{\sim}{\omega}}{\partial t} + \underline{u}\cdot\nabla\underset{\sim}{\omega} = (\underset{\sim}{\omega}\cdot\nabla)\underline{u}$$

$$\frac{\partial\omega_\theta}{\partial t}\hat{\underline{e}}_\theta + \underline{u}\cdot\nabla(\omega_\theta\hat{\underline{e}}_\theta) = \omega_\theta\frac{1}{R}\frac{\partial}{\partial\theta}(u_R\hat{\underline{e}}_R + u_z\hat{\underline{e}}_z)$$

using $\frac{\partial}{\partial \theta}\hat{\underset{\sim}{e}}_R = \hat{e}_\theta$

$$\frac{\partial \omega_\theta}{\partial t}\hat{\underset{\sim}{e}}_\theta + \underset{\sim}{u}\cdot\nabla(\omega_\theta \hat{\underset{\sim}{e}}_\theta) = \frac{\omega_\theta}{R} u_R \hat{\underset{\sim}{e}}_\theta$$

using $\frac{\partial}{\partial R}\hat{\underset{\sim}{e}}_\theta = 0$, $\frac{\partial}{\partial z}\hat{\underset{\sim}{e}}_\theta = 0$ and $u_R = \underset{\sim}{u}\cdot\nabla R$

Divide through by R:

$$\frac{\partial}{\partial t}\left(\frac{\omega_\theta}{R}\right)\hat{\underset{\sim}{e}}_\theta + \underbrace{\frac{1}{R}\underset{\sim}{u}\cdot\nabla\omega_\theta \hat{\underset{\sim}{e}}_\theta - \hat{e}_\theta\frac{\omega_\theta}{R^2}\underset{\sim}{u}\cdot\nabla R}_{\underset{\sim}{u}\cdot\nabla\left(\frac{\omega_\theta}{R}\right)\hat{\underset{\sim}{e}}_\theta} = 0$$

Therefore

$$\frac{\partial}{\partial t}\left(\frac{\omega_\theta}{R}\right) + \underset{\sim}{u}\cdot\nabla\left(\frac{\omega_\theta}{R}\right) = 0$$

or

$$\underline{\underline{\frac{D}{Dt}\left(\frac{\omega_\theta}{R}\right) = 0}}$$

Hence the vorticity of a smoke ring moves with the fluid, and changes in proportion to R.

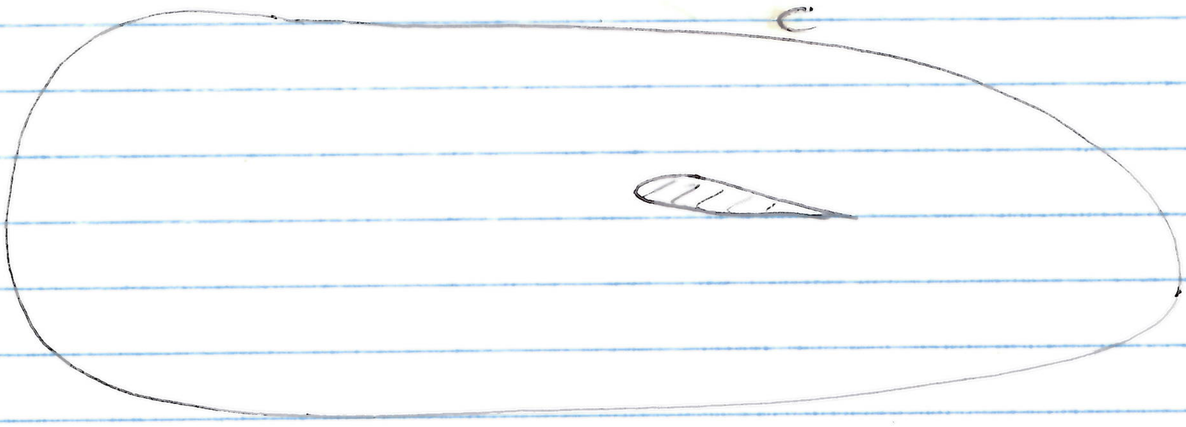
Starting vortex

Reminder: Kelvin's theorem: In the absence of viscosity, if $C(t)$ is a circuit which consists of the same fluid particles as time proceeds, then

$$\Gamma = \int_{C(t)} \underline{u} \cdot dx \quad \text{is independent of time}$$

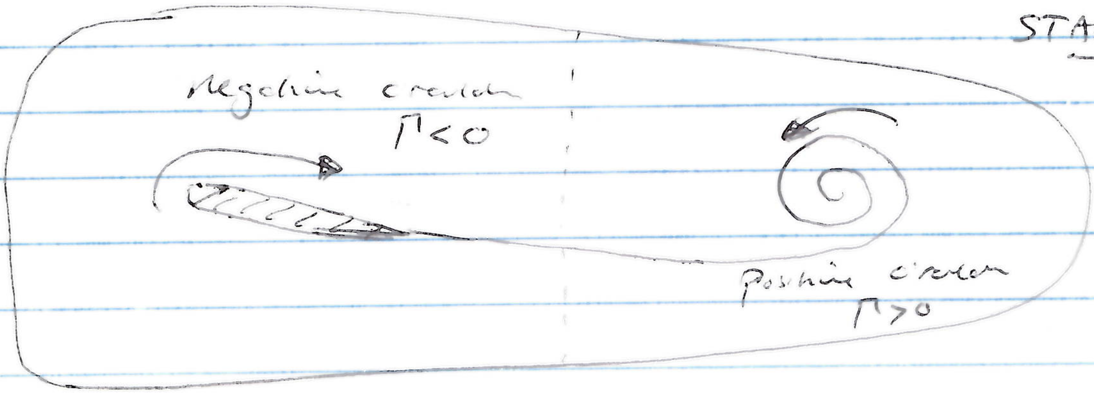
Note: Does not depend on incompressibility, does not require surface to be only within the fluid

Consider a wing initially at rest $\Gamma = \int_{C(t)} \underline{u} \cdot dx = 0$



How does the negative circulation around the wing occur?

STARTING VORTEX



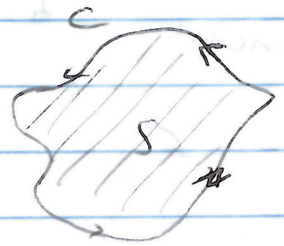
By Kelvin's theorem, if there is a negative circulation around the wing, then there must have been a vortex shed from the wing with positive circulation.

10.5

3D Wings

Stoke's theorem says:

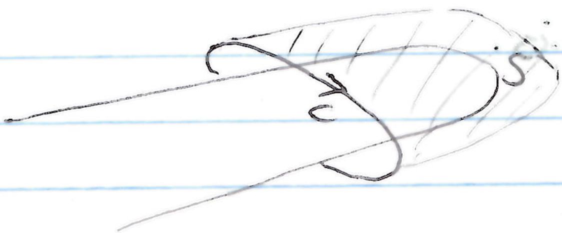
$$\oint_C \underline{u} \cdot d\underline{x} = \int_S (\nabla \times \underline{u}) \cdot \underline{n} \, dS$$



- In 2D ~~flow~~ irrotational flow $\nabla \times \underline{u} = 0$ and so

  $\Gamma = \oint \underline{u} \cdot d\underline{x} = 0$ except where the surface encloses the wing.

- In 3D we can always create a surface which lies entirely in the fluid.

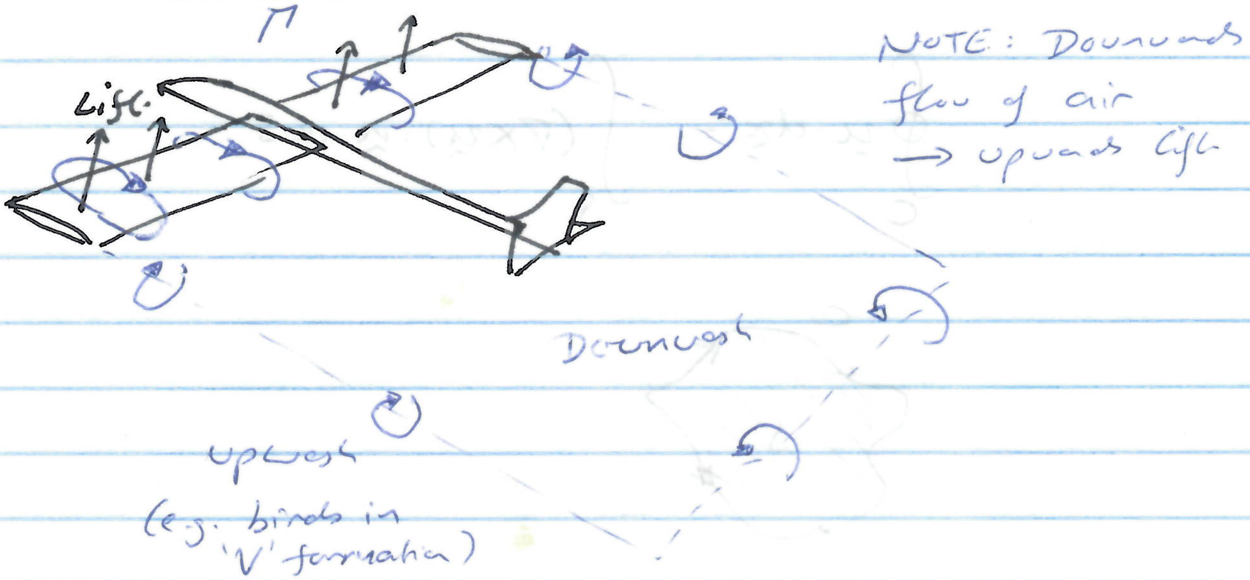


From Stoke's theorem, if there is any circulation (and so lift) then there must be vorticity $\nabla \times u \neq 0$

$\Rightarrow$ This is seen as a Trailing vortex

10.5

The combination of a starting vortex and trailing vortex produces a 'horseshoe vortex' behind an aircraft in 3D



In reality the lift generated varies along the length of the wing. Whenever the circulation (lift) varies, vorticity is shed from the wing

(lifting line theory)

Shedding of vorticity is necessary in 3D for lift. The aircraft must do work ~~on~~ the air → induced drag even when no viscosity.