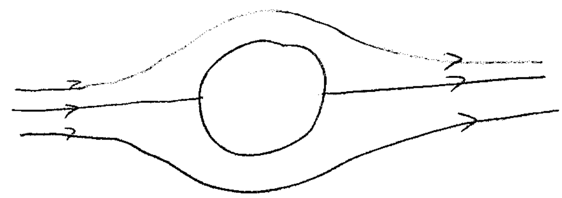
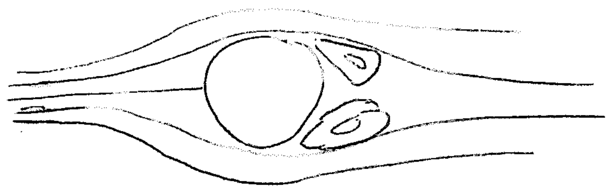
Lecture 11

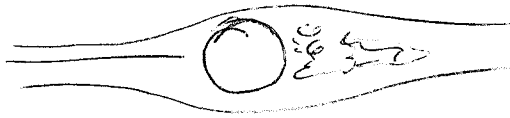
Inviscid (Ideal) flow around a cylinder



Real flow around a cylinder looks more like



or

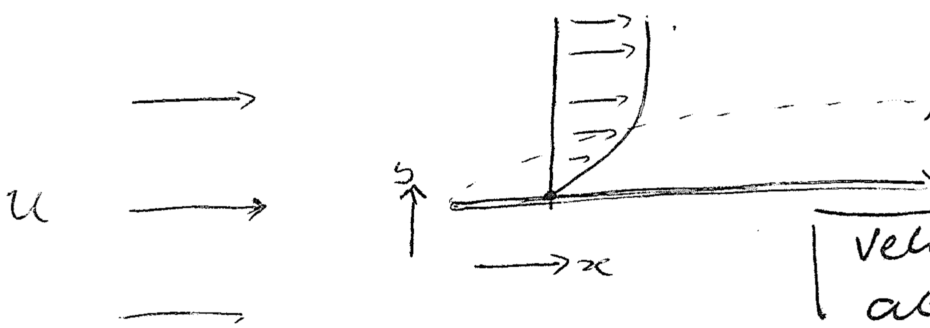


Trailing edge very different from ideal theory

What is the cause of this?

- It is an effect of the viscosity of the fluid. Close to the surface a boundary layer forms in which viscosity is important regardless of how small the viscosity is.

Flow past a flat plate



$\boxed{\text{Velocity goes to zero at the surface}}$

Ignoring pressure gradients (for now)

Momentum equation $\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = \nu \nabla^2 \underline{u}$

NOTE: $\nu = \frac{\mu}{\rho}$

ν - kinematic viscosity
μ - dynamic viscosity

steady flow $\frac{\partial \underline{u}}{\partial t} = 0$

$$\left.\begin{aligned} \underline{u} \cdot \nabla \underline{u} &\simeq U \frac{1}{x} U \\ \nu \nabla^2 u &\simeq \nu \frac{1}{\delta^2} U \end{aligned}\right] \quad U^2 \frac{1}{x} \sim \nu \frac{1}{\delta^2} U$$

$$\delta^2 \approx \frac{\nu x}{U}$$

$$\boxed{\delta \sim \sqrt{\frac{\nu x}{U}}}$$

As viscosity becomes smaller, the boundary layer becomes thinner

Note: for a body of scale L $(\sim x)$

$$\frac{\delta}{L} \sim \sqrt{\frac{\nu}{UL}} = \sqrt{\frac{1}{Re}} \quad \text{Reynolds number}$$

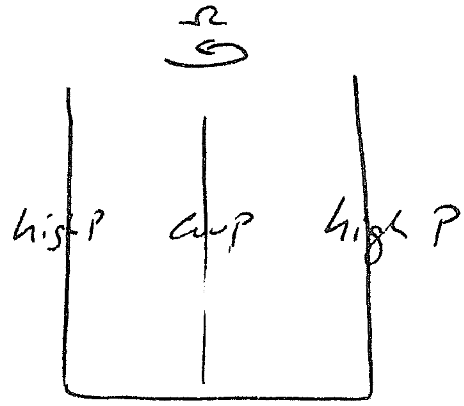
If Reynolds number is small, then the boundary layer is comparable to the size of the body
- Viscosity important everywhere, not a "layer"

At high Reynolds number a thin boundary layer forms, in which viscosity is important, whilst the flow away from the surface is approximately ideal.

11.2

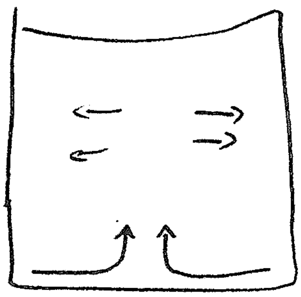
Because the boundary layer is thin, it can usually be assumed that the pressure is constant across the layer.

Example: spin-down of a stirred cup of tea

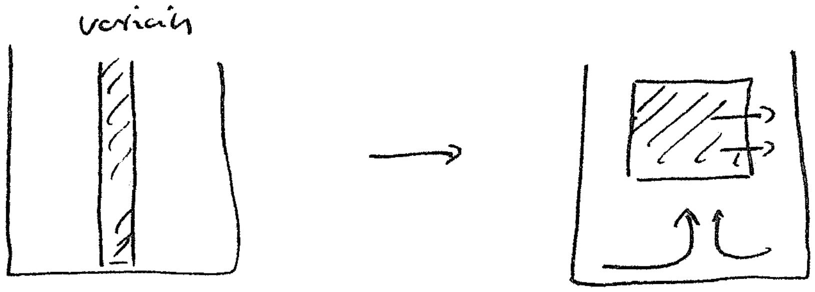


In the bulk of the fluid the centrifugal force is balanced by a pressure gradient

close to the bottom of the cup the boundary layer slows the rotation, but the pressure ~~is~~ still has a radial gradient.
→ causes an inwards flow



This is an example of vortex compression



By Kelvin's theorem this leads to a spin-down

~~Hex~~

The friction ~~can be estimated by integrating the momentum equation over the boundary layer~~ is given by the gradient

$$F \approx \int_0^{\delta} \mu \nabla^2 u \, dy \sim \rho \nu \frac{1}{\delta^2} u \times \delta$$

$\mu |\nabla u| \quad \frac{\partial u}{\partial y}$

so the force per unit ~~width~~ area on the flat plate varies as

$$F \sim \rho \nu \sqrt{\frac{U}{\nu x}}\, U = \rho \sqrt{\frac{\nu U}{x}}\, U$$

Based on $x \to L$ (length of plate), force per unit ~~width~~ area

$$F \sim \rho \underbrace{\sqrt{\frac{\nu}{UL}}}_{Re^{-1/2}} U^2$$

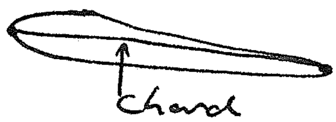
Typically define a drag coefficient

$$C_D = \frac{F_x}{\frac{1}{2}\rho U^2 S}$$

so for flat plate $C_D \sim Re^{-1/2}$

(numerical solution $C_D \sim 1.3\, Re^{-1/2}$) Blasius' solution

wing plan ~~surface~~ area - projection onto the chord of all sections



compare lift coefficient

$$C_L = \frac{F_y}{\frac{1}{2}\rho U^2 S}$$

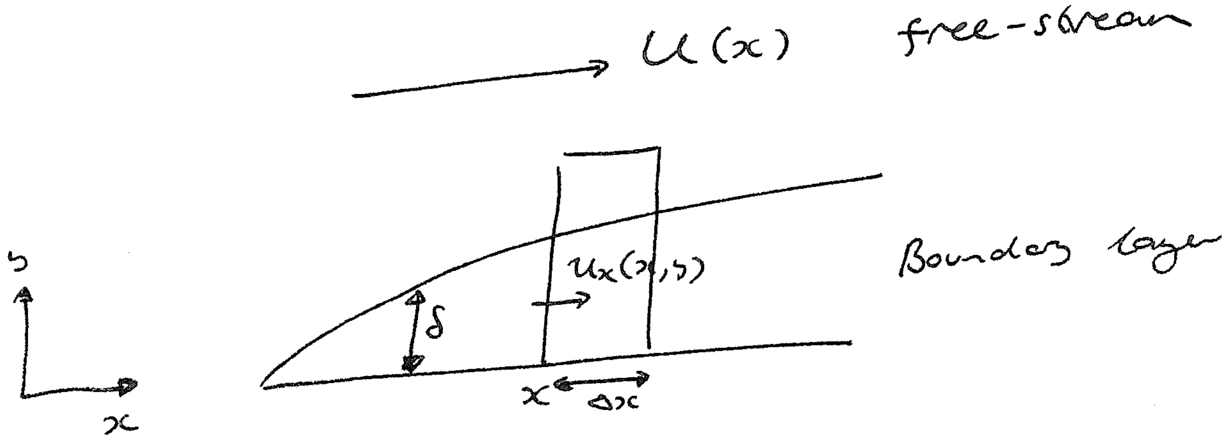
e.g. thin wing $C_L = \dfrac{\pi \rho U^2 L \sin\alpha}{\frac{1}{2}\rho U^2 L} \simeq 2\pi\alpha$

11.3

Von Karman's Momentum Integral Method

For 2D flow this method provides a (relatively) straightforward method to calculate the friction due to a boundary layer

Consider flow over a flat plate



Starting from the momentum equation

$$\rho\left[\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right] = -\nabla p + \mu\nabla^2\underline{u}$$

steady state, $\frac{\partial \underline{u}}{\partial t} = 0$

Take the x component (along surface)

$$\rho\left[u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\nabla^2 u_x \qquad (1)$$

The boundary layer is thin in y, long in x

$$\Rightarrow \nabla^2 u_x \simeq \frac{\partial^2 u_x}{\partial y^2}$$

We are going to make some assumptions about the velocity profile

Boundary conditions

$u_x = 0$ at $y = 0$ — * correct boundary at a surface

$u_x = U(x)$ at $y = \delta$ — Matches free flow

$\dfrac{\partial u_x}{\partial y} = 0$ at $y = \delta$ — smooth, no friction

$\dfrac{\partial^2 u_x}{\partial y^2} = 0$ at $y = 0$ — Since $u_x \simeq 0$ close to the wall.

The flow is incompressible, so $\nabla \cdot \underset{\sim}{u} = 0$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

and so, multiplying by $\rho(U - u_x)$ for reasons which will become clear shortly:

$$\rho(U - u_x)\frac{\partial u_x}{\partial x} + \rho(U - u_x)\frac{\partial u_y}{\partial y} = 0 \qquad (2)$$

In the free flow the pressure is given by Bernoulli's theorem. Assuming that the boundary layer is thin, $\frac{\partial P}{\partial y} \simeq 0$

and so
$$P + \tfrac{1}{2}\rho U^2 = \text{const}$$

$$\text{so} \quad \frac{\partial P}{\partial x} + \rho U \frac{\partial U}{\partial x} = 0 \qquad (3)$$

11.4

putting ① and ③ together:

$$\rho\left[U\frac{dU}{dx} - u_x\frac{\partial u_x}{\partial x} - u_y\frac{\partial u_x}{\partial y}\right] = -\mu\frac{\partial^2 u_x}{\partial y^2}$$

Prandtl's Boundary Layer Approximation

and adding ② to the left-hand side

$$\rho\left[\underbrace{U\frac{dU}{dx}}_{①} + \underbrace{U\frac{\partial u_x}{\partial x} - u_x\frac{\partial u_x}{\partial x}}_{②} \underbrace{- u_x\frac{\partial u_x}{\partial x}}_{①} + \underbrace{U\frac{\partial u_y}{\partial y} - u_x\frac{\partial u_y}{\partial y}}_{②} \underbrace{- u_y\frac{\partial u_x}{\partial y}}_{①}\right] = -\mu\frac{\partial^2 u_x}{\partial y^2}$$

$$U\frac{\partial u_x}{\partial x} \to \frac{\partial}{\partial x}(Uu_x) - u_x\frac{\partial U}{\partial x}$$

$$- u_x\frac{\partial u_x}{\partial x} - u_x\frac{\partial u_x}{\partial x} \to -\frac{\partial}{\partial x}(u_x^2)$$

$$U\frac{\partial u_y}{\partial y} \to \frac{\partial}{\partial y}(Uu_y)$$

$$- u_x\frac{\partial u_y}{\partial y} - u_y\frac{\partial u_x}{\partial y} \to -\frac{\partial}{\partial y}(u_x u_y)$$

$$\rho\left[U\frac{\partial U}{\partial x} - u_x\frac{\partial U}{\partial x} + \frac{\partial}{\partial x}(Uu_x - u_x^2) + \frac{\partial}{\partial y}(Uu_y - u_x u_y)\right] = -\mu\frac{\partial^2 u_x}{\partial y^2}$$

Now integrate from $y=0$ to $y=\delta$

$$\rho\frac{\partial U}{\partial x}\int_0^\delta (U-u_x)\,dy + \rho\frac{\partial}{\partial x}\int_0^\delta (Uu_x - u_x^2)\,dy + \underbrace{\int_0^\delta \frac{\partial}{\partial y}(Uu_y - u_x u_y)\,dy}_{\left[u_y(U-u_x)\right]_0^\delta} = -\mu \underbrace{\int_0^\delta \frac{\partial^2 u_x}{\partial y^2}\,dy}_{\left[\frac{\partial u_x}{\partial y}\right]_0^\delta}$$

This $\to 0$ because at $y=\delta$, $u_x = U$ and $y=0$ $u_y=0$. This is why ② has this form.

$$\left[\frac{\partial u_x}{\partial y}\right]_0^\delta = -\left.\frac{\partial u_x}{\partial y}\right|_{y=0}$$

because $\frac{\partial u_x}{\partial y} = 0$ at $y=\delta$

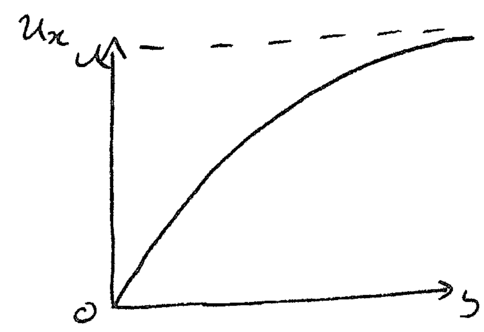
Therefore

$$\rho \frac{\partial U}{\partial x}\int_0^\delta (U - u_x)\,dy + \rho \frac{\partial}{\partial x}\int_0^\delta (U u_x - u_x^2)\,dy = \mu \left.\frac{\partial u_x}{\partial y}\right|_{y=0}$$

This can be solved by assuming a form for u_x and substituting into this equation.

The simplest form which satisfies our boundary condition is

$$u_x = U\left[\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]$$



over a flat plate $\frac{\partial U}{\partial x} = 0$ so we have

$$\rho \frac{\partial}{\partial x}\int_0^\delta \left[U^2\left(\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right) - U^2\left(\frac{9}{4}\left(\frac{y}{\delta}\right)^2 - \frac{3}{2}\left(\frac{y}{\delta}\right)^4 + \frac{1}{4}\left(\frac{y}{\delta}\right)^6\right)\right]$$

$$= \mu \left.\frac{\partial u_x}{\partial y}\right|_{y=0} = \mu \frac{3}{2}\frac{U}{\delta}$$

$$U^2 \rho \frac{\partial}{\partial x}\underbrace{\left[\frac{3}{4}\frac{y^2}{\delta} - \frac{1}{8}\frac{y^4}{\delta^3} - \frac{9}{12}\frac{y^3}{\delta^2} + \frac{3}{10}\frac{y^5}{\delta^4} - \frac{1}{28}\frac{y^7}{\delta^6}\right]_0^\delta}_{\frac{39}{280}\delta} = \mu \frac{3}{2}\frac{U}{\delta}$$

so $\frac{39}{280}\rho U^2 \frac{d\delta}{dx} = \frac{3}{2}\mu \frac{U}{\delta}$

Integrating, setting $\delta = 0$ at $x = 0$ gives

$$\boxed{\delta = 4.64\sqrt{\frac{\mu x}{\rho U}}}$$

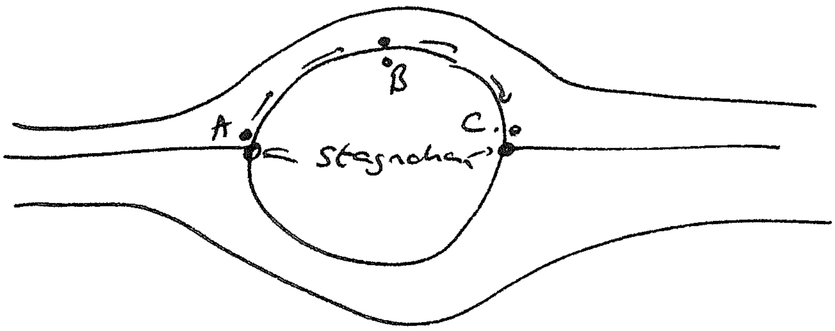
from $F = \mu \left.\frac{\partial u_x}{\partial y}\right|_{y=0}$

$C_D = 1.29\,Re^{-1/2}$

11.5

Boundary layers have important effects on flow around objects such as aerofoils and cylinders.

Without viscosity we have ideal irrotational flow



Consider a fluid particle moving along the flow close to the surface. from $A \to B \to C$.

The pressure is a maximum at A (the stagnation point), falls to a minimum at B. This pressure gradient accelerates the fluid, so that Bernoulli's relation holds

$$p + \frac{1}{2}\rho u^2 = \text{const.}$$

Once at B, the pressure starts increasing toward another maximum at C. This <u>adverse pressure gradient</u> (pressure increasing in the direction of flow) slows down the flow speed to zero again.

Viscosity (Boundary layer) slows down the fluid flow from $A \to B \to C$, and so rather than reaching zero speed at C, the fluid is brought to rest at some point between B and C.

When this happens the flow <u>separates</u>.

Flow separation