Lecture 13

Compressible flow

So far we have assumed that the fluid is incompressible, so $\nabla\cdot\underline{u}=0$ and $\rho=$ constant. Now we will relax this constraint.

Our equations are

$$\rho\left(\frac{\partial \underline{u}}{\partial t}+\underline{u}\cdot\nabla\underline{u}\right)=-\nabla p \qquad \text{Euler's equation}$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\underline{u})=0 \qquad \text{Mass conservation}$$

Now we need an equation of state to link pressure and density. In this course we will assume adiabatic gas:

$$\frac{P}{\rho^{\gamma}}=\text{constant}$$

γ - Ratio of specific heats

($\simeq 1.4$ for air)

(also called isentropic approximation)

using $\nabla\left(\frac{P}{\rho^{\gamma}}\right)=0$

$$\nabla\left(\frac{P}{\rho^{\gamma}}\right)=\frac{1}{\rho^{\gamma}}\nabla P+\frac{P\gamma}{\rho^{\gamma-1}}\nabla\left(\frac{1}{\rho}\right)=0$$

or $\frac{1}{\rho}\nabla P+P\gamma\nabla\left(\frac{1}{\rho}\right)=0$ or $P\nabla\left(\frac{1}{\rho}\right)=-\frac{1}{\gamma\rho}\nabla P$

Similarly

$$\nabla\left(\frac{P}{\rho}\right)=\frac{1}{\rho}\nabla P+P\nabla\left(\frac{1}{\rho}\right)=\frac{1}{\rho}\nabla P\left(\frac{\gamma-1}{\gamma}\right)$$

Hence $\quad \frac{1}{\rho}\nabla p = \nabla\left(\frac{\gamma}{\gamma-1}\frac{p}{\rho}\right)$

For steady flows we have

$$\underbrace{\underline{u}\cdot\nabla\underline{u}}_{\nabla\left(\frac{1}{2}u^2\right)} = \underbrace{-\frac{1}{\rho}\nabla p}_{\nabla\left(\frac{\gamma}{\gamma-1}\frac{p}{\rho}\right)}$$

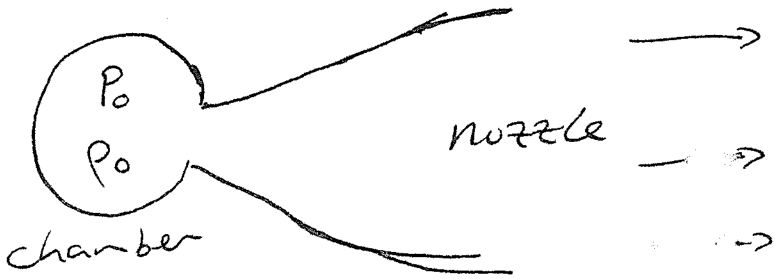
and steady compressible flow:

$$\nabla\left(\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho}\right) = 0$$

i.e. $$\boxed{\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \text{constant}}$$

Bernoulli's equation for compressible flow

First we are going to look at steady flows of a compressible gas : <u>Rocket engines</u>



Fuel is burnt in a chamber sustaining a pressure p_0 and density ρ_0

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The purpose of a rocket engine is to propel the exhaust at as high a speed as possible, in order to generate thrust.

What is the maximum speed the exhaust gas can reach?

Bernoulli's equation

$$\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{P}{\rho} = \text{constant}$$

Since $u \simeq 0$ inside the combustion chamber, and exhaust gas flows along a streamline from the combustion chamber

$$\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{P}{\rho} = \frac{\gamma}{\gamma-1}\frac{P_0}{\rho_0}$$

Since P, ρ and u^2 are all positive, the maximum value of u^2 occurs when $P = 0$

So $$\frac{1}{2}u_{max}^2 = \frac{\gamma}{\gamma-1}\frac{P_0}{\rho_0}$$

$$u_{max} = \sqrt{\frac{2\gamma}{\gamma-1}\frac{P_0}{\rho_0}}$$

Example: Oxygen-hydrogen rocket

Combustion temperature ~ 3,500 K

chamber pressure ~ 100 Atm (10^7 Pa)

$$P = n k_B T$$

so $$\rho = \frac{P m_a}{k_B T}$$ $$\frac{P_0}{\rho_0} = \frac{k_B T}{m_a}$$

average molecular weight

$\gamma \sim 1.2$ ~ 13 (mixture ratio 5-6)

So $\rho = 4.5 \ \mathrm{kg/m^3}$

$$u_{max} = 5.2 \ \mathrm{km/s}$$

<u>Mass flux density</u>

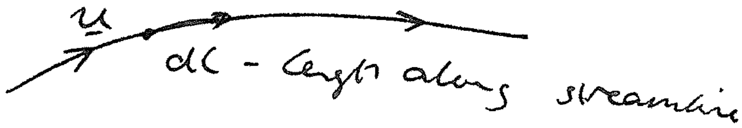
The mass flux density $J = \rho u$ [$\mathrm{kg/m^2/s}$] varies along a streamline (through the nozzle)

The change in flux density is:

$$d(\rho u) = \rho\, du + u\, d\rho$$

We can then use Euler's equation in steady state:

$$\underbrace{\underline{u} \cdot \nabla \underline{u}}_{\text{Derivative of } u \text{ along a streamline}} = -\frac{1}{\rho} \nabla P$$



Along a streamline (in the direction of flow)

$$u \frac{du}{dl} = -\frac{1}{\rho} \frac{dP}{dl}$$

and since l is the only independent parameter,

$$u\, du = -\frac{1}{\rho}\, dP$$

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Using the adiabatic equation of state

$$\frac{p}{\rho^\gamma} = \text{constant} = k$$

$$\frac{dp}{d\rho} = \frac{k\gamma\rho^\gamma}{\rho} = \frac{\gamma p}{\rho} = c^2 \qquad c = \text{speed of sound}$$

$$\text{So} \quad dp = c^2 d\rho$$

$$u\,du = -\frac{1}{\rho}c^2 d\rho \qquad \rightarrow \quad d\rho = -\rho u/c^2\, du$$

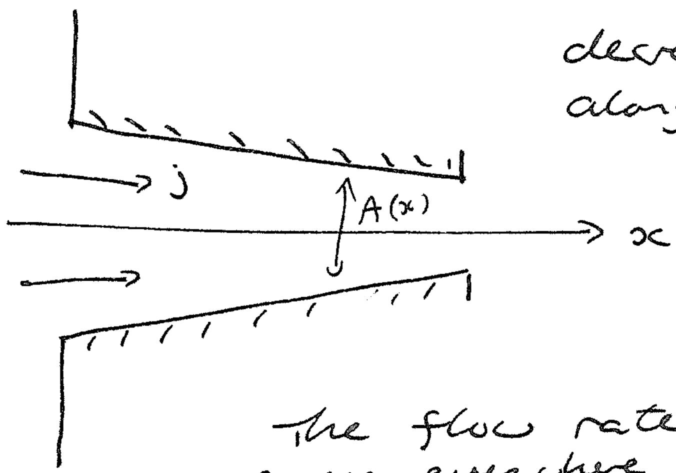
$$d(\rho u) = \rho\,du + u\left[-\rho u/c^2\, du\right]$$

$$\frac{d(\rho u)}{du} = \rho\left(1 - u^2/c^2\right)$$

The maximum flux density occurs when the ~~flow~~ speed u is equal to the local sound speed, c.

<u>Example</u>: converging nozzle

The area $A(x)$ decreases with distance along the flow, x.

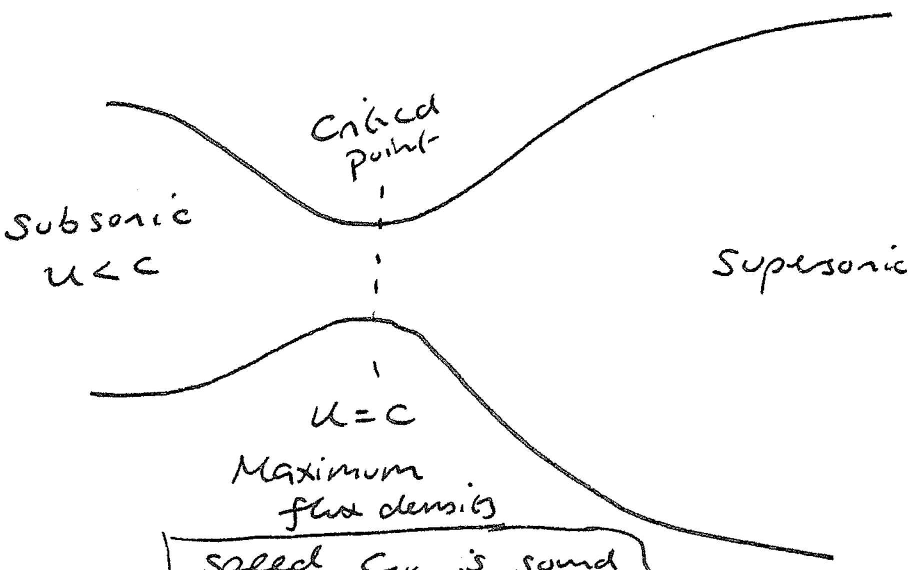


The flow rate must be the same everywhere, so $jA = \text{const}$

Since $jA = \text{const}$, and A is a minimum at the end of the converging nozzle, j must be a maximum at the end of the nozzle

Therefore: It is impossible to get a transition to supersonic flow in a converging channel. The maximum flow velocity possible is sound speed at the end (narrowest point)

This is why rocket exhaust 'de Laval' nozzles first contract then expand.



$\boxed{\text{speed } c_* \text{ is sound speed at this critical point}}$

Using $jA = \rho u A = \text{const} = \Gamma$

$$d(\rho u) = d\left(\frac{\Gamma}{A}\right) = -\frac{\Gamma}{A^2}\,dA = -\frac{\rho u}{A}\,dA$$

So $$-\frac{\rho u}{A}\frac{dA}{du} = \rho\left(1 - u^2/c^2\right)$$

$$\underline{\frac{du}{u} = \frac{1}{u^2/c^2 - 1}\,\frac{dA}{A}}$$

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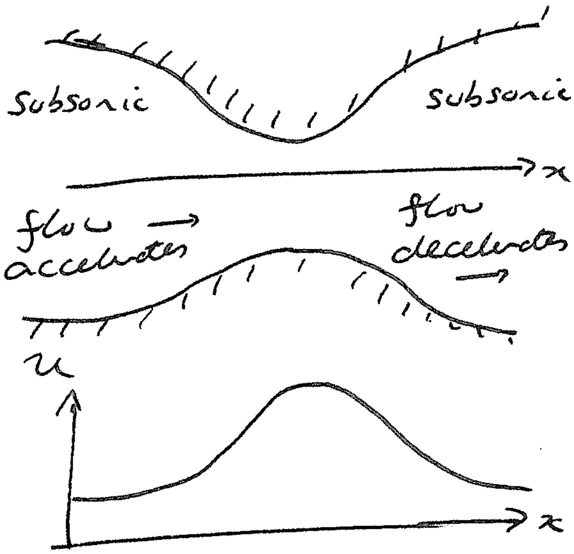
If the flow is subsonic, $u < c$, then this equation says that as the cross-section gets smaller ($dA < 0$) then the speed increases ($du > 0$), and vise-versa

Sub-sonic flow:



If the flow rate is too low then the fluid never reaches the local sound speed. As the nozzle expands, the flow slows down.

As the pressure inside is increased, or the pressure outside is decreased, the mass flux increases until it reaches the maximum value

$$J_* = \rho_* c_*$$

local sound speed at critical point

local density at critical point.

The flow is now 'choked'. At this point the mass flow does not depend on the pressure outside the rocket, though increasing chamber pressure (increasing ρ) do still increase flow rate.

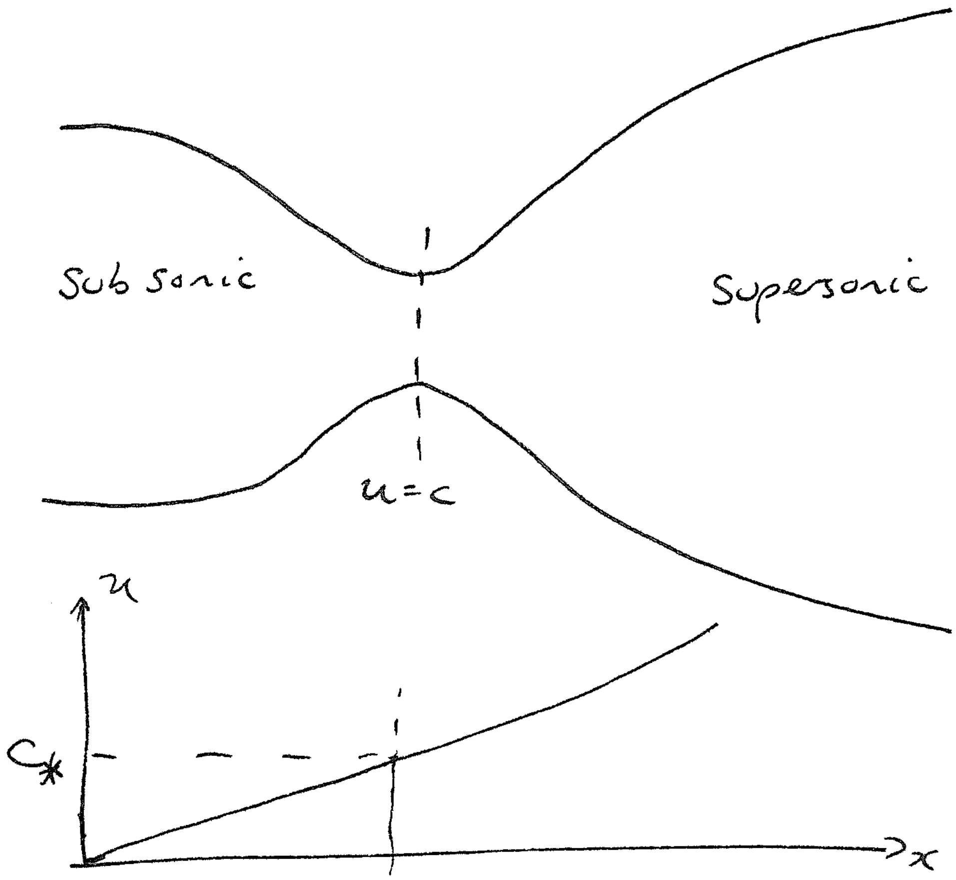
when the flow is choked it reaches $u = c$ at the narrowest point (throat) (Mach 1).

when flow becomes supersonic ($u > c$) the relationship between A and u reverses:

$$\frac{du}{u} = \underbrace{\frac{1}{M^2 - 1}}_{>0} \frac{dA}{A}$$

$M = u/c$ Mach number

As the area A increases, the flow speed now increases rather than decreases



So how do we calculate c_*, ρ_* etc.?

Bernoulli eqⁿ for compressible gas:

$$\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{P}{\rho} = \text{constant} = \frac{\gamma P_0}{(\gamma-1)\rho_0}$$

At the critical point $u = c_*$ and $c_* = \sqrt{\frac{\gamma P}{\rho}}$

$$\text{So} \quad \frac{1}{2}c_*^2 + \frac{c_*^2}{\gamma-1} = \frac{\gamma}{\gamma-1}\frac{P_0}{\rho_0}$$

$$\text{so} \quad c_*^2\left[1 + \frac{(\gamma-1)}{2}\right] = \frac{\gamma P_0}{\rho_0}$$

$$c_* = \sqrt{\frac{\gamma P_0}{\rho_0}}\sqrt{\frac{2}{\gamma+1}} \qquad (1)$$

$\gamma \simeq 1.2$, so $\quad c_* \simeq 0.95\sqrt{\frac{\gamma P_0}{\rho_0}}$

for the hydrogen-oxygen rocket example

$$c_* = 1.55 \text{ km/s}$$

The density can be calculated from the equation of state $\quad \frac{P}{\rho^\gamma} = \text{const}$

So the values at the critical point, P_* and ρ_* are related to the chamber values P_0 and ρ_0:

$$\frac{P_*}{\rho_*^\gamma} = \frac{P_0}{\rho_0^\gamma} \qquad \text{So} \quad P_* = P_0\left(\frac{\rho_*}{\rho_0}\right)^\gamma \qquad (2)$$

By definition $c_*^2 = \dfrac{\gamma P_*}{\rho_*}$, the local sound speed

From the previous result, 1

$$c_*^2 = \frac{\gamma P_0}{\rho_0} \frac{2}{\gamma+1} = \frac{\gamma P_*}{\rho_*}$$

Substituting eq^n 2 for P_* we get

$$\frac{\gamma P_0}{\rho_0} \frac{2}{\gamma+1} = \frac{\gamma}{\rho_*} P_0 \left(\frac{\rho_*}{\rho_0}\right)^{\gamma}$$

$$\text{so} \quad \frac{2}{\gamma+1} = \left(\frac{\rho_*}{\rho_0}\right)^{\gamma-1}$$

$$\underline{\rho_* = \rho_0 \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}} \qquad (3)$$

Our hydrogen-oxygen example had $\rho_0 = 4.5\ \mathrm{kg/m^3}$ and $\gamma \simeq 1.2$ so

$$\rho_* \simeq 2.8\ \mathrm{kg/m^3}$$

Hence the maximum flux density is

$$J_* = c_* \rho_* = 4.3 \times 10^3\ \mathrm{kg/m^2/s}$$

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Example: what are the dimensions of a rocket engine producing 7×10^6 N (Saturn V F-1) burning hydrogen-oxygen at 3,500 K and 100 atm?

Assume exhaust into vacuum, so we reach u_{max} and $P=0$ at exit

$$\text{Force} = \Gamma V_{exhaust}$$

$\uparrow$ flow rate $\rho u A$ [kg/s]

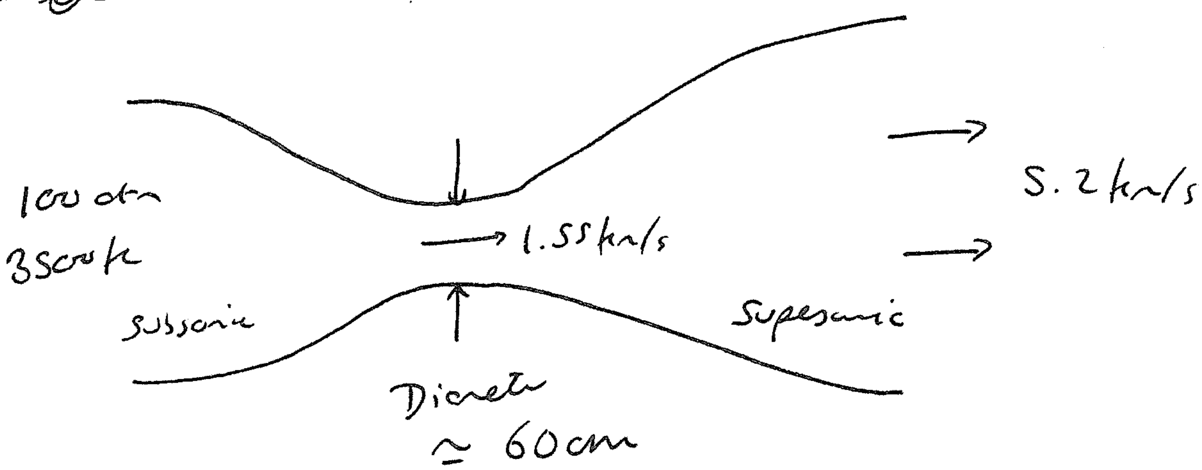
from earlier $u_{max} = \sqrt{\frac{2\gamma}{\gamma-1}\frac{P_0}{\rho_0}} \simeq 5.2$ km/s

So to produce 7×10^6 N mass flow of $\Gamma = 1.4\times10^3$ kg/s

The maximum flux is approx. 4.3×10^3 kg/m²/s at the throat, so the area of the throat

$$A_{throat} \simeq \frac{1.4\times10^3}{4.3\times10^3} \simeq 0.3\ m^2$$

~~The maximum area (at the end of the exhaust) can be calculated~~



$$\left[\text{power} = \tfrac{1}{2}\Gamma V^2 \simeq 1.6\ \text{GW}\right]$$

using Bernoulli

$$\tfrac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{P}{\rho} = \text{const} = \frac{\gamma}{\gamma-1}\frac{P_0}{\rho_0}$$

the fact that the flow rate is constant

$$u\rho A = \Gamma \quad \text{(constant)}$$

Write $$u = \frac{c_* \rho_* A_*}{\rho A} \quad \text{(flux through throat)}$$

to give:

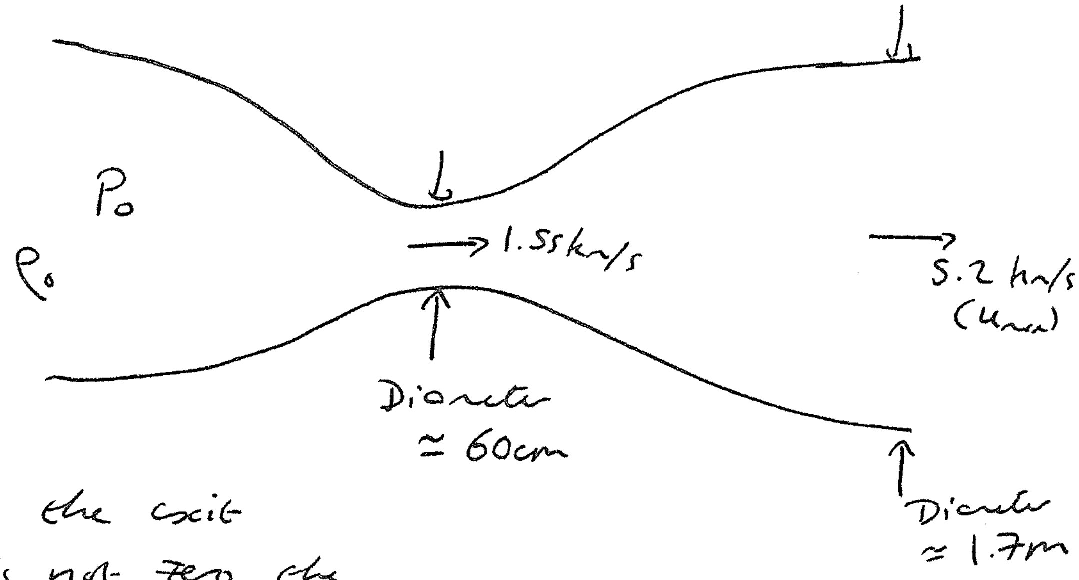
$$\frac{A}{A_*} = \left(\frac{\gamma-1}{\gamma+1}\right)^{1/2}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\left[\left(\frac{P}{P_0}\right)^{\frac{\gamma+1}{\gamma}} + \left(\frac{P}{P_0}\right)^{2/\gamma}\right]^{-1/2}$$

At the end of the nozzle the pressure should match ambient pressure

e.g. to operate at sea level ($1\,\text{atm} = 10^5\,\text{Pa}$)

$P/P_0 \simeq 10^{-2}$ $\qquad \gamma \simeq 1.2$

so $A/A_* \simeq 7.2$ and so the exit diameter $\simeq 1.7\,\text{m}$



Note: Since the exit pressure is not zero, the final velocity will not quite be u_{max}.