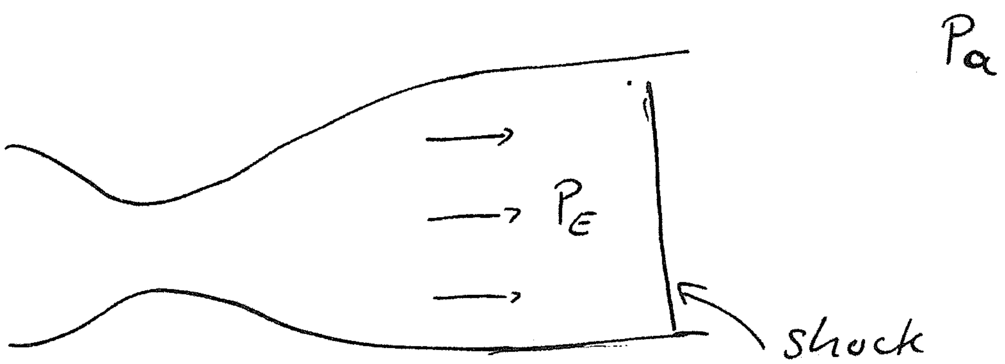
Lecture 14

Shocks

So far we have studied expanding compressible flows (Rocket engines). Now we will look at compression, which is associated with shocks.

This can occur in a rocket engine exhaust if the pressure of the exhaust is lower than ambient (over-expanded) or higher than ambient (under-expanded)

e.g. over-expanded

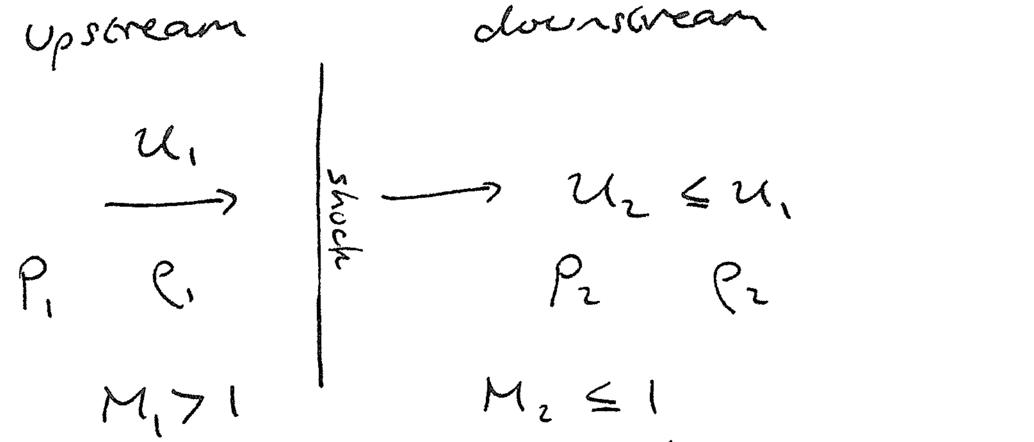


Exhaust pressure lower than ambient ($P_E < P_a$)
→ a shock forms where the fluid slows to subsonic, and pressure increases.

As we'll see, shocks form when fluid is compressed, not when it expands: there is no shock in the throat

What is a shock? It is a discontinuous change in the fluid flow e.g. velocity, pressure, density. In real fluids the thickness of these regions is ~ the mean free path

e.g. air at sea level $\lambda \simeq 70\,\mathrm{nm}$



frame where shock is stationary

Even though the flow is discontinuous, and so in general we can't take derivatives across a shock, we can still apply conservation principles

- Mass conservation
- Momentum conservation
- Energy conservation

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Mass conservation implies that

$$\rho_1 u_1 = \rho_2 u_2$$

The momentum (Euler's) equation can be written as

$$\frac{\partial(\rho \underline{u})}{\partial t} + \nabla\cdot\left(p\underline{\underline{I}} + \rho\underline{u}\,\underline{u}\right) = 0$$

or in 1D

$$\frac{\partial}{\partial t}(\rho u_x) + \cancel{\nabla}\frac{\partial}{\partial x}\big(\underbrace{P + \rho u_x^2}_{\text{Momentum flux}}\big) = 0$$

→ continuous

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$$

Energy conservation means that the flux of energy is constant:

$$\underbrace{\tfrac{1}{2}\rho u^2}_{\substack{\text{kinetic energy}\\ \text{per volume}}} + h\rho$$

~~En~~ Internal energy per mass (specific enthalpy)

$$h = \frac{\gamma}{\gamma-1}\frac{P}{\rho}$$

for adiabatic fluids

flux of energy is

$$\left(\tfrac{1}{2}\rho u^2 + h\rho\right)u$$

so $\left(\tfrac{1}{2}u_1^2 + h_1\right)\rho_1 u_1 = \left(\tfrac{1}{2}u_2^2 + h_2\right)\rho_2 u_2$

and since $\rho_1 u_1 = \rho_2 u_2$, energy conservation gives:

$$\frac{1}{2} u_1^2 + \frac{\gamma}{\gamma-1} \frac{P_1}{\rho_1} = \frac{1}{2} u_2^2 + \frac{\gamma}{\gamma-1} \frac{P_2}{\rho_2}$$

<u>Bernoulli's equation</u>

So we have three equations:

$$\boxed{\begin{aligned} \rho_1 u_1 &= \rho_2 u_2 \qquad (1) \\ P_1 + \rho_1 u_1^2 &= P_2 + \rho_2 u_2^2 \qquad (2) \\ \frac{1}{2} u_1^2 + \frac{\gamma}{\gamma-1} \frac{P_1}{\rho_1} &= \frac{1}{2} u_2^2 + \frac{\gamma}{\gamma-1} \frac{P_2}{\rho_2} \qquad (3) \end{aligned}}$$

RANKINE-HUGONIOT equations
(for adiabatic fluid $h = \frac{\gamma}{\gamma-1} \frac{P}{\rho}$)

This set of 3 equations gives a unique solution for the downstream flow (P_2, ρ_2, u_2) in terms of the upstream flow (P_1, ρ_1, u_1)

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from ②

$$P_1 - P_2 + \rho_1 u_1^2 - \underbrace{\rho_2 u_2}_{\rho_1 u_1} u_2 = 0$$

so $$P_1 - P_2 + \rho_1 u_1 (u_1 - u_2) = 0$$

Rearranging ③

$$\frac{\gamma}{\gamma-1}\left(\frac{P_1}{\rho_1} - \frac{P_2}{\rho_2}\right) + \frac{1}{2}\underbrace{(u_1^2 - u_2^2)}_{(u_1+u_2)(u_1-u_2)} = 0$$

using
$$P_2 = P_1 + \rho_1 u_1 (u_1 - u_2)$$
and $$\rho_2 = \rho_1 \frac{u_1}{u_2}$$

$$\frac{\gamma}{\gamma-1}\left(\frac{P_1}{\rho_1} - \frac{u_2}{\rho_1 u_1}\left[P_1 + \rho_1 u_1 (u_1 - u_2)\right]\right) + \frac{1}{2}(u_1+u_2)(u_1-u_2) = 0$$

$$\frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1}\left(1 - \frac{u_2}{u_1}\right) - \frac{\gamma}{\gamma-1} u_2 (u_1 - u_2) + \frac{1}{2}(u_1+u_2)(u_1-u_2) = 0$$

so $$\left[\frac{P_1}{\rho_1 u_1} - u_2 + \frac{\gamma-1}{2\gamma}(u_1+u_2)\right](u_1 - u_2) = 0$$

so one of these brackets is zero: Either the trivial solution $u_1 = u_2$ (no shock) or

$$\frac{P_1}{\rho_1 u_1} - u_2 + \frac{\gamma-1}{2\gamma}(u_1+u_2) = 0$$

Defining the upstream Mach number

$$M_1 = u_1/c_1 = \frac{u_1}{\sqrt{\gamma P_1/\rho_1}}$$

So $$\frac{P_1}{\rho_1 u_1} = \frac{u_1}{\gamma M_1^2}$$

$$\frac{u_1}{M_1^2} - \gamma u_2 + \frac{(\gamma-1)}{2}(u_1+u_2) = 0$$

$$\frac{u_1}{u_2} - \gamma M_1^2 + \frac{(\gamma-1)}{2}\left(\frac{u_1}{u_2}+1\right)M_1^2 = 0$$

$$\frac{u_1}{u_2}\left[1+\frac{(\gamma-1)}{2}M_1^2\right] = \gamma M_1^2 - \frac{(\gamma-1)}{2}M_1^2$$

$$\frac{u_1}{u_2}\frac{1}{2}\left[2+(\gamma-1)M_1^2\right] = \frac{1}{2}(\gamma+1)M_1^2$$

so $$\boxed{\frac{u_1}{u_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}}$$

since $\frac{u_1}{u_2} = \frac{\rho_2}{\rho_1}$, this is the density compression ratio

Note that as the shock gets stronger (faster) $M_1 \to \infty$ then $\frac{\rho_2}{\rho_1} \to \frac{(\gamma+1)}{(\gamma-1)}$

So for $\gamma = 5/3$ $\frac{\rho_2}{\rho_1} \to 4$ $\underset{\text{(air)}}{\gamma \simeq 1.4}$ $\frac{\rho_2}{\rho_1} \to 6$

Hence there is a limit to the density increase which can be achieved with a single shock. This has implications in areas such as inertial fusion, where a large compression ratio can be achieved with multiple shocks.

The pressure ratio can be calculated from ②

$$1 + \frac{\rho_1 u_1^2}{P_1} = \frac{P_2}{P_1} + \frac{\rho_2 u_2^2}{P_1}$$

using $\rho_2 = \rho_1 \frac{u_1}{u_2}$ and $\frac{\rho_1 u_1^2}{P_1} = \gamma M_1^2$

$$1 + \gamma M_1^2 = \frac{P_2}{P_1} + \gamma M_1^2 \frac{u_2}{u_1}$$

Substituting previous result for $\frac{u_2}{u_1}$ and simplifying:

$$\boxed{\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - \gamma + 1}{(\gamma+1)}}$$

So as the upstream Mach number increases the pressure ratio keeps increasing

We can also work out the downstream Mach number

$$M_2^2 = \frac{u_2^2}{c_2^2} = \frac{u_2^2 \rho_2}{\gamma P_2} = \underbrace{\frac{u_1^2 \rho_1}{\gamma P_1}}_{M_1^2} \frac{\rho_1}{\rho_2} \frac{P_1}{P_2}$$

$$= M_1^2 \left[\frac{(\gamma-1)M_1^2 + 2}{(\gamma+1)M_1^2} \right] \left[\frac{\gamma+1}{2\gamma M_1 - \gamma + 1} \right]$$

$$= \frac{(\gamma-1)M_1^2 + 2}{2\gamma M_1^2 - \gamma + 1} = \frac{(\gamma-1)M_1^2 + 2}{(\gamma-1)M_1^2 + 2 + (M_1^2 - 1)(\gamma+1)}$$

Since $M_1 > 1$ (supersonic into shock)

$\Rightarrow M_2^2 < 1$ (subsonic after shock)

To show that we <u>cannot</u> have a shock where $M_1 < 1$, $M_2 > 1$ (i.e. in expansion) we can calculate the change in entropy

$$S = C_V \ln\left(\frac{P}{\rho^\gamma}\right) + \text{const} \qquad [\text{Perfect gas}]$$

$$\Delta S = C_V \left[\ln\left(\frac{P_2}{\rho_2^\gamma}\right) - \ln\left(\frac{P_1}{\rho_1^\gamma}\right) \right]$$

$$= C_V \left\{ \ln\left(\frac{P_2}{P_1} \cdot \frac{\rho_1^\gamma}{\rho_2^\gamma}\right) \right\} = C_V \left[\ln\left(\frac{P_2}{P_1}\right) - \gamma \ln\left(\frac{\rho_2}{\rho_1}\right) \right]$$

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Hence

$$\Delta S = C_v \ln\left[\frac{2\gamma M_1^2 - \gamma + 1}{\gamma + 1}\right] - C_v \gamma \left[\frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2}\right]$$

Taking the derivation of ΔS w.r.t M_1:

$$\frac{d\Delta S}{dM_1} = C_v \frac{4\gamma M_1}{2\gamma M_1^2 - \gamma + 1} - \frac{C_v \gamma}{M_1}\left[\frac{4}{(\gamma-1)M_1^2 + 2}\right]$$

$M_1 = 1$

$$\frac{d\Delta S}{dM_1} = C_v\left[\frac{4\gamma}{2\gamma - \gamma + 1} - \frac{4\gamma}{(\gamma-1)+2}\right] = 0$$

$$\frac{d^2\Delta S}{dM_1^2} = -C_v \frac{4\gamma(2\gamma M_1^2 + \gamma - 1)}{(\gamma(2M_1^2 - 1) + 1)^2} + C_v \frac{4\gamma[3(\gamma-1)M_1^2 + 2]}{M_1^2[(\gamma-1)M_1^2 + 2]^2}$$

$M_1 = 1$

$$\frac{d^2\Delta S}{dM_1^2} = -C_v \frac{4\gamma(3\gamma - 1)}{(\gamma+1)^2} + C_v \frac{4\gamma(3\gamma - 1)}{(\gamma+1)^2} = 0$$

$$\frac{d^3\Delta S}{dM_1^3} = C_v \frac{16\gamma^2 M_1[\gamma(2M_1^2 + 3) - 3]}{[\gamma(2M_1^2 - 1) + 1]^3}$$

$$- \frac{16 C_v \gamma[3(\gamma-1)^2 M_1^4 + 3(\gamma-1)M_1^2 + 2]}{M_1^3[(\gamma-1)M_1^2 + 2]^3}$$

$M_1 = 1$

$$\frac{d^3\Delta S}{dM_1^3} = C_v \frac{16\gamma^2[5\gamma - 3]}{(\gamma+1)^3} - \frac{16 C_v \gamma[3\gamma^2 - 3\gamma + 2]}{(\gamma+1)^3}$$

$$\frac{d^3 \Delta s}{dM_1^3} = \frac{32 c_v \gamma (\gamma^2 - 1)}{(\gamma+1)^3}$$

Since $\gamma > 1$, $\frac{d^3 \Delta s}{dM_1^3} > 0$

Therefore

- If $M_1 > 1$ $\Delta S > 0$ and entropy increases
- If $M_1 < 1$ $\Delta S < 0$ entropy decreases
  $\rightarrow$ Forbidden!

Shocks can only occur in compression