Lecture 7

$pV^{\alpha} = \text{Const}$

Polytropic gas
Adiabatic
ideal ?

isobaric $\alpha = 0$
isothermal $\alpha = 1$
adiabatic $\alpha = \gamma$

- Review types of flow

Incompressible $\quad \nabla \cdot \underline{u} = 0$

Irrotational $\quad \nabla \times \underline{u} = 0$

- Derivative along the flow $\quad \dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + \underline{u} \cdot \nabla$

Momentum equation

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u} = -\nabla\left(\frac{P}{\rho} + \chi\right)$$

$\chi = gz$ gravitational potential

so $\underline{g} = -\nabla \chi$

Identity: $(\underline{u} \cdot \nabla)\underline{u} = (\nabla \times \underline{u}) \times \underline{u} + \nabla\left(\frac{1}{2}u^2\right)$

$$\Rightarrow \quad \frac{\partial \underline{u}}{\partial t} + (\nabla \times \underline{u}) \times \underline{u} = -\nabla \underbrace{\left(\frac{P}{\rho} + \frac{1}{2}u^2 + \chi\right)}_{H}$$

steady flow

$$(\nabla \times \underline{u}) \times \underline{u} = -\nabla H$$

Dot with u:

$$(\underline{u} \cdot \nabla) H = 0 \qquad \text{i.e. } H \text{ constant along flow}$$

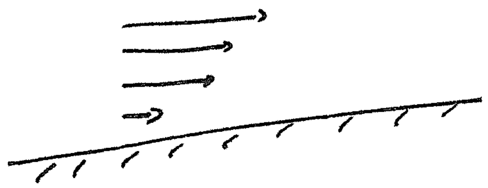
Bernoulli streamline theorem

If $\nabla \times \underline{u} = 0$ then $\nabla H = 0$ i.e. H constant

* idea of vorticity $\nabla \times \underline{u} = \underline{\omega}$

Vorticity: <u>local</u> rotation of fluid elements

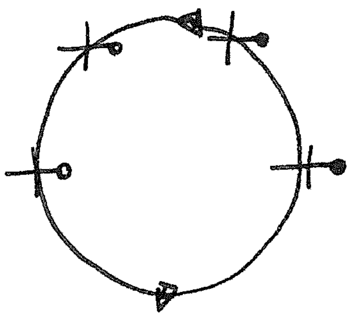
e.g.



$$\underset{\sim}{u} = (\beta y, 0, 0)$$

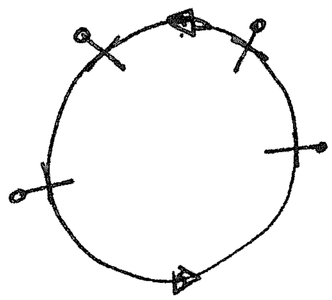
$$\Rightarrow \quad \underset{\sim}{\omega} = -\beta \hat{z}$$

no global rotation, but has vorticity



$u_\theta \propto \frac{1}{r}$

no vorticity

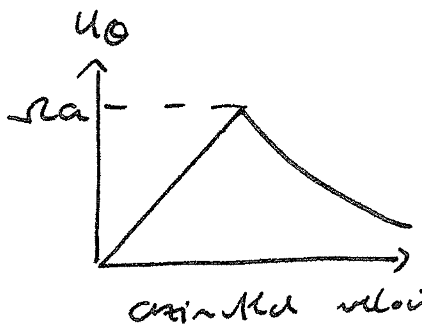


$u_\theta \propto r$

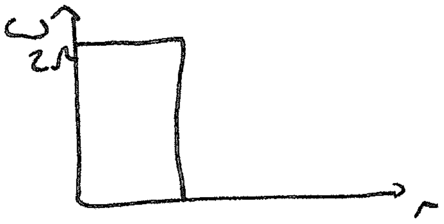
has vorticity

Rankine vortex

$$u_\theta = \begin{cases} \Omega r & r < a \\ \dfrac{\Omega a^2}{r} & r > a \end{cases}$$



azimuthal velocity



vorticity

Steady flow past a wing at small angles of incidence is typically irrotational

7.2

Proof: The vorticity equation

$$\frac{\partial \underline{u}}{\partial t} + \underset{\sim}{\omega} \times \underline{u} = -\nabla H$$

Take curl

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} + \nabla \times (\underset{\sim}{\omega} \times \underset{\sim}{u}) = 0$$

NB: Pressure removed

Vector identities

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} + (\underline{u}\cdot\nabla)\underset{\sim}{\omega} - (\underset{\sim}{\omega}\cdot\nabla)\underline{u} + \underset{\sim}{\omega}\,\underbrace{\cancel{\nabla\cdot\underline{u}}}_{\text{Incompress.}} - \underline{u}\,\underbrace{\cancel{\nabla\cdot\underset{\sim}{\omega}}}_{\substack{0 \\ \nabla\cdot\nabla\times = 0}} = 0$$

$$\Rightarrow \quad \frac{\partial \underset{\sim}{\omega}}{\partial t} + (\underline{u}\cdot\nabla)\omega = (\underset{\sim}{\omega}\cdot\nabla)\underline{u}$$

$$\boxed{\frac{D\underset{\sim}{\omega}}{Dt} = (\underset{\sim}{\omega}\cdot\nabla)\underline{u}}$$

In 2D flow

$$\underline{u} = [\,u_x(x,y,\cancel{z},t),\ u_y(x,y,\cancel{z},t),\ 0\,]$$

$$\underset{\sim}{\omega} = [\ 0,\ 0,\ \omega_z\,]$$

so $(\underset{\sim}{\omega}\cdot\nabla)\underline{u} = \omega_z \dfrac{\partial \underline{u}}{\partial z} = 0$

and so $\boxed{\dfrac{D\omega}{Dt} = 0}$

In 2D flow of an ideal incompressible fluid the vorticity of each fluid element is conserved

steady flow $(\underline{u}\cdot\nabla)\omega_z = 0$

If $\underset{\sim}{\omega} = \nabla \times \underset{\sim}{v} = 0$ everywhere then the flow is <u>irrotational</u>. In this case $\nabla H = 0$

and so

$$H = \frac{P}{\rho} + \frac{1}{2}u^2 + \chi$$

is a constant everywhere.

Irrotational flow allows the velocity to be written as a potential ϕ:

$$\underset{\sim}{v} = \nabla \phi \qquad (\text{so } \nabla \times \underset{\sim}{v} = \nabla \times \nabla \phi \equiv 0)$$

Greatly simplifies analysis, played significant role in development of aerodynamics.

However,

1. This flow is dissipationless (ideal) so doesn't create friction, drag forces

2. As we shall see, irrotational flow is not unique. It allows arbitrary discontinuities at the surface of a body

3. In 2D the flow is only unique if a circulation is also provided by ad-hoc assumptions (e.g. Kutta-Joukowski Condition)