Lecture 8 Flow around a cylinder

Assuming irrotational flow $\nabla \times \underline{v} = 0$

so using potential

$$V = \nabla \phi$$

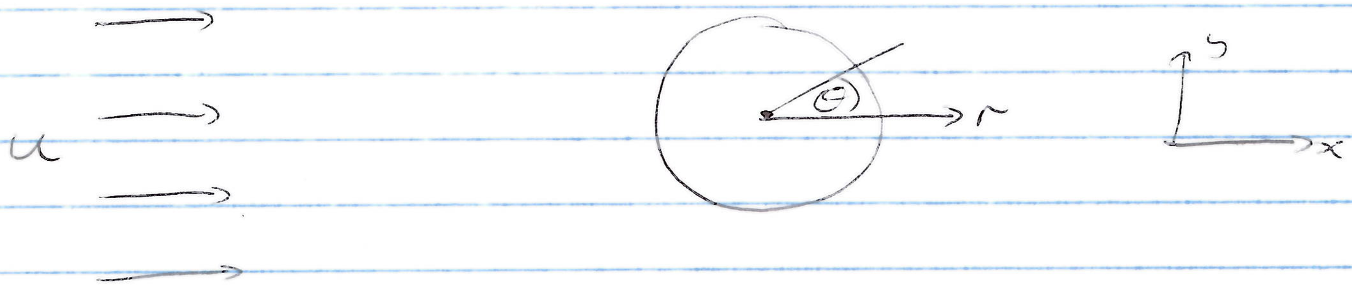
~~In addition assume~~

Since flow is incompressible $\nabla \cdot \underline{v} = 0$

so

$$\underline{\nabla^2 \phi = 0} \qquad \text{(Laplace's equation)}$$

Consider a cylinder in a uniform flow



In polar coordinates

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Boundary conditions

(1) No flow through surface so

$$\frac{\partial \phi}{\partial r} = 0 \quad \text{at} \quad r = a$$

(2) Velocity $\to U$ in x direction as ~~x~~ $\to \pm\infty$

$$\phi \to Ur\cos\theta \quad \text{as} \quad r \to \pm\infty$$

8.2

Look for $\phi = \phi_r(r)\,\phi_\theta(\theta)$

$$\phi_\theta\left(\frac{\partial^2\phi_r}{\partial r^2} + \frac{1}{r}\frac{\partial\phi_r}{\partial r}\right) + \phi_r\frac{1}{r^2}\frac{\partial^2\phi_\theta}{\partial\theta^2} = 0$$

$$\frac{r^2}{\phi_r}\left(\frac{\partial^2\phi_r}{\partial r^2} + \frac{1}{r}\frac{\partial\phi_r}{\partial r}\right) = -\frac{1}{\phi_\theta}\frac{\partial^2\phi_\theta}{\partial\theta^2} = \lambda$$

Must be a constant since LHS only $f(r)$ and RHS $f(\theta)$

$\Rightarrow$ θ dependence

$$\frac{\partial^2\phi_\theta}{\partial\theta^2} = -\lambda\phi_\theta \qquad \phi_\theta = A\sin(\lambda^{1/2}\theta) + B\cos(\lambda^{1/2}\theta)$$

To match boundary at $r = \infty$, only $\cos\theta$ term, $\lambda = 1$

$$\Rightarrow \quad \phi_\theta = B\cos\theta$$

r dependence

$$r^2\frac{\partial^2\phi_r}{\partial r^2} + r\frac{\partial\phi_r}{\partial r} - \phi_r = 0$$

try $\phi_r = Ar^n$

$$r^2 A\,n(n-1)r^{n-2} + r\,nAr^{n-1} - Ar^n = 0$$

$$\Rightarrow n(n-1) + n - 1 = 0 \qquad \underline{n = 1 \text{ or } -1}$$

$$\Rightarrow \quad \phi_r = Ar + \frac{C}{r}$$

As $r \to \infty$ $\quad \phi_r \to Ar = Ur \quad \Rightarrow \quad \underline{A = U}$

at $r = a$ $\quad \frac{\partial\phi_r}{\partial r} = 0 \quad \Rightarrow \quad A - \frac{C}{a^2} = 0 \quad \Rightarrow \quad C = \underline{\underline{a^2 U}}$

So the solution is

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta$$

However we can add to this

(1) a constant, but this has no effect $(v = \nabla\phi)$

(2) a linear function of θ

$$\phi = \frac{\Gamma\theta}{2\pi} \qquad \Gamma = \text{circulation}$$

$$\boxed{\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta + \frac{\Gamma\theta}{2\pi}}$$

Not unique, requires knowledge of circulation either from

(1) Ad-hoc assumption, used shortly

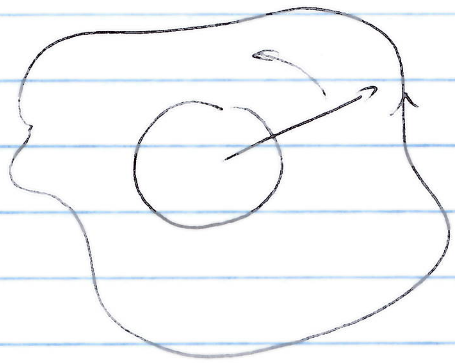
(2) Viscous boundary effects not included in irrotational flow theory.

What effect does this circulation have?

→ will show that this is the origin of lift, via the Kutta-Joukowski theory.

Why is Γ circulation?

$$\Gamma = \oint_c \underline{V} \cdot dl = \underbrace{\oint_c \nabla\phi \cdot dl}_{\text{change in } \phi \text{ after one circuit}}$$



θ changes by 2π

ϕ changes by Γ

8.3 Applying Bernoulli's theorem to flow around a cylinder

$$p + \tfrac{1}{2}\rho u^2 = \text{const} \quad \text{on cylinder surface}$$

since it's a streamline

$u_r = 0$ at $r = a$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}$$

Hence at $r = a$ $\quad = -2U\sin\theta + \dfrac{\Gamma}{2\pi a}$

$$\Rightarrow u^2 = u_\theta^2 = 4U^2\sin^2\theta - \frac{2U\Gamma}{\pi a}\sin\theta + \frac{\Gamma^2}{4\pi^2 a^2}$$

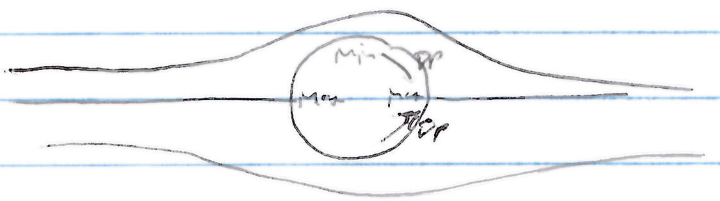
Bernoulli

$$p + \tfrac{1}{2}\rho\left[4U^2\sin^2\theta - \frac{2U\Gamma}{\pi a}\sin\theta + \frac{\Gamma^2}{4\pi^2 a^2}\right] = \text{const}$$

constant $\Rightarrow$ ignore

Note: (1) pressure is fore-aft symmetric $\theta \to \pi - \theta$
so no drag force (d'Alembert's "paradox")

(2) pressure is a maximum at the stagnation points



* Increasing pressure in direction of flow can lead to separation of boundary layer - see later in course.

If we take the y component of the pressure force on the cylinder

$$F_y = \oint -p \sin\theta \underbrace{a\, d\theta}_{\text{area}}$$

$$= \rho \oint \left[2u^2 \sin^2\theta - \frac{u\Gamma}{\pi a} \sin\theta \right] \sin\theta \, d\theta$$

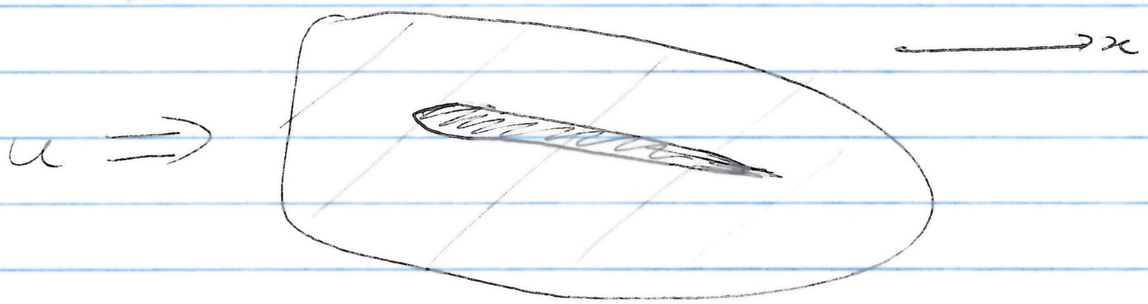
odd in θ → average is zero

$$= \underline{\underline{-\rho u \Gamma}} \qquad \text{since} \qquad \oint \sin^2\theta \, d\theta = \pi$$

This is a specific case of the much more general Kutta-Joukowski theorem

$$\boxed{\begin{array}{l}\text{For irrotational, 2D flow around } \underline{\text{any}} \text{ shaped body } \cancel{\text{is go}} \text{ the lift force is given by} \\ \qquad\qquad F_y = -\rho u \Gamma\end{array}}$$

Kutta-Joukowski lift formula (Handout?)



Momentum flux $\quad F_i = \oint \underbrace{(p\delta_{ij} + \rho v_i v_j)}_{\text{Momentum flux}} dS_j$

perturbed x velocity $\quad \delta v_x = v_x - U$

$\rightarrow$ Bernoulli $\quad p \simeq p_0 - \rho U \delta v_x$

Total mass flux must be zero

$$\oint \rho (v_x dS_x + v_y dS_y) = \oint \rho(\delta v_x dS_x + \delta v_y dS_y) = 0$$

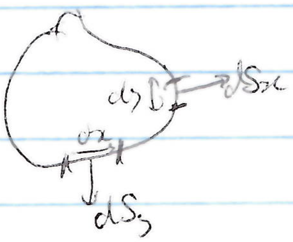
Taking the y component of F:

$$F_y = -\oint \rho v_y v_x dS_x + (p + \rho v_y v_y) dS_y$$

$$= -\oint \rho \underbrace{v_y v_x}_{U\delta v_y \text{ - linearize}} dS_x + (p_0 - \rho U \delta v_x + \underbrace{\rho v_y v_y}_{\text{ignore}}) dS_y$$

$$\simeq -\oint p_0 dS_y - \rho U \oint (\delta v_y dS_x - \delta v_x dS_y)$$

In 2D



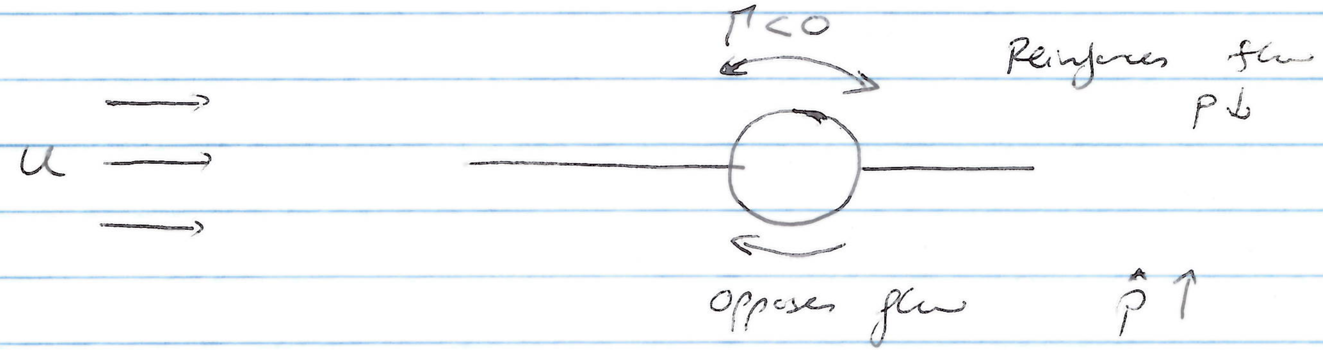
$dS_x = dy$

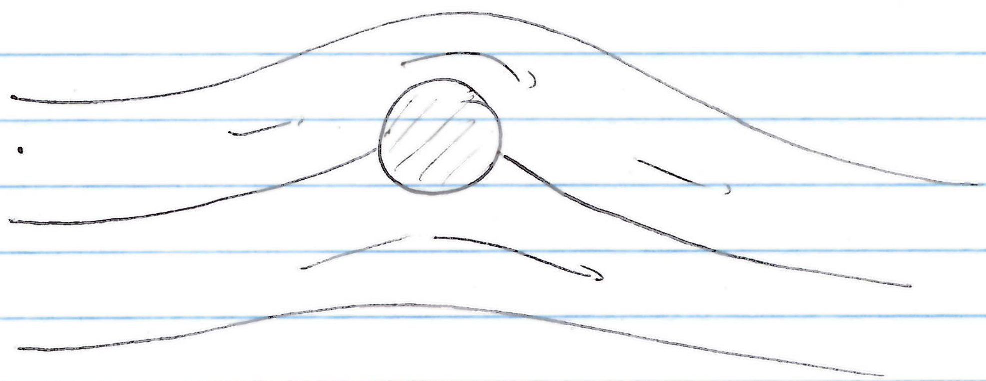
$dS_y = -dx$

$$\Rightarrow F_y = -\rho U \overbrace{\oint (\delta v_y dy + \delta v_x dx)}^{\Gamma}$$

8.5

What does (negative) circulation do to the flow





Downwards flow of momentum

$\Rightarrow$ upwards push on body

What can cause $\Gamma \neq 0$?

1. Rotation of body, friction and separation of boundary (not irrotational!)
   → Magnus effect

2. Asymmetric shape w.r.t. oncoming flow
   → Aerofoils!