Neoclassical transport

Dr Ben Dudson

Department of Physics, University of York
Heslington, York YO10 5DD, UK

28^{th} January 2013

Last time

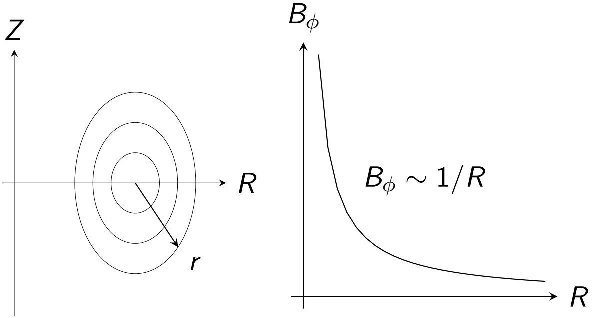
- Toroidal devices such as the Stellarator and Tokamak
- Need for a rotational transform to short out the vertical electric field caused by the ∇B drift
- This can either be created using shaped coils (Stellarators) or by running a current in the plasma (Tokamaks)

Last time

- Toroidal devices such as the Stellarator and Tokamak
- Need for a rotational transform to short out the vertical electric field caused by the ∇B drift
- This can either be created using shaped coils (Stellarators) or by running a current in the plasma (Tokamaks)
- Calculated the "classical" transport of particles and energy out of a tokamak. This gave a wildly unrealistic confinement
- This lecture we'll look at one reason why...

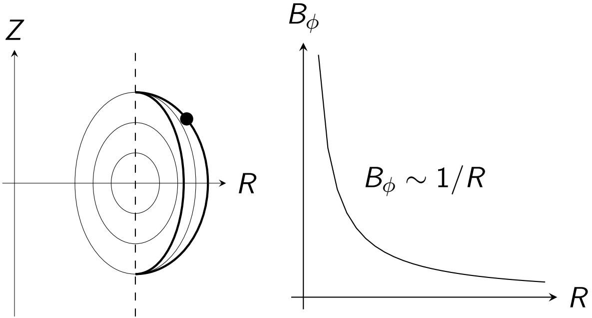
Particle trapping

In a tokamak, the magnetic field varies with the major radius



Particle trapping

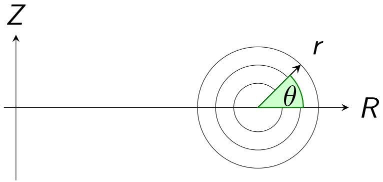
In a tokamak, the magnetic field varies with the major radius



There is now a minimum in the B field at the outboard (large R) side of the tokamak ⇒ Trapped particles.

The study of these particles and their effect is called NEOCLASSICAL THEORY.

A useful approximation is that the variation in major radius R is small, and that the toroidal field is much bigger than the poloidal field.



For a circular cross-section of radius r, the major radius varies like

$$R = R_0 + r\cos\theta = R_0\left(1 + \epsilon\cos\theta\right)$$

$\boxed{\epsilon = r/R_0}$ is **inverse aspect ratio**

Hence

$$B \simeq \frac{B_0}{1 + \epsilon\cos\theta}$$

For small $\epsilon \ll 1$

$$B \simeq B_0\left(1 - \epsilon\cos\theta\right)$$

Particle trapping (redux)

- If there is no electrostatic potential ϕ, Kinetic energy $\frac{1}{2}mv^2$ is conserved (i.e. speed is conserved)
- Magnetic magnetic moment also conserved $\mu = mv_\perp^2/2B$

- If there is no electrostatic potential ϕ, Kinetic energy $\frac{1}{2}mv^2$ is conserved (i.e. speed is conserved)
- Magnetic magnetic moment also conserved $\mu = mv_\perp^2/2B$

Define velocity at the outboard side ($\theta = 0$): $v_{\perp 0}$ and $v_{||0}$.
K.E. conserved

$$v^2 = v_{||}^2 + v_\perp^2 = v_{||0}^2 + v_{\perp 0}^2 \qquad \Rightarrow v_{||}^2 = v^2\left(1 - \frac{v_\perp^2}{v^2}\right)$$

Conservation of μ

$$\frac{v_\perp^2}{B_0\left(1 - \epsilon\cos\theta\right)} = \frac{v_{\perp 0}^2}{B_0\left(1 - \epsilon\right)}$$

$$v_{||}^2 = v^2\left(1 - \frac{v_{\perp 0}^2}{v^2}\frac{1 - \epsilon\cos\theta}{1 - \epsilon}\right) = v^2\left(1 - \frac{v_{\perp 0}^2}{v^2}\left[1 + \epsilon\left(1 - \cos\theta\right)\right]\right)$$

Particle trapping (redux)

- If there is no electrostatic potential ϕ, Kinetic energy $\frac{1}{2}mv^2$ is conserved (i.e. speed is conserved)
- Magnetic magnetic moment also conserved $\mu = mv_\perp^2/2B$

Define velocity at the outboard side ($\theta = 0$): $v_{\perp 0}$ and $v_{||0}$.
K.E. conserved

$$v^2 = v_{||}^2 + v_\perp^2 = v_{||0}^2 + v_{\perp 0}^2 \qquad \Rightarrow v_{||}^2 = v^2\left(1 - \frac{v_\perp^2}{v^2}\right)$$

Conservation of μ

$$\frac{v_\perp^2}{B_0\left(1 - \epsilon\cos\theta\right)} = \frac{v_{\perp 0}^2}{B_0\left(1 - \epsilon\right)}$$

$$v_{||}^2 = v^2\left(1 - \frac{v_{\perp 0}^2}{v^2}\frac{1 - \epsilon\cos\theta}{1 - \epsilon}\right) = v^2\left(1 - \frac{v_{\perp 0}^2}{v^2}\left[1 + \epsilon\left(1 - \cos\theta\right)\right]\right)$$

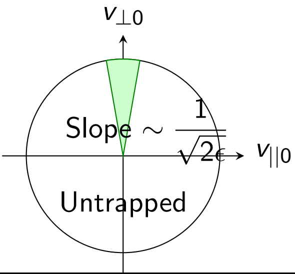
$$v_{||}^2 = v^2\left(1 - \frac{v_{\perp 0}^2}{v^2}\left[1 + 2\epsilon\sin^2\left(\theta/2\right)\right]\right)$$

$$v_{||}^2 = v^2\left(1 - \frac{v_{\perp 0}^2}{v^2}\left[1 + 2\epsilon\sin^2\left(\theta/2\right)\right]\right)$$

If $v_{||}^2 < 0$ for any θ then a particle is trapped. Therefore,

$$\text{if} \quad v_{||}^2\left(\theta = \pi\right) \leq 0 \quad \Rightarrow \text{Particle is trapped}$$

$$\frac{v_{\perp 0}^2}{v^2}\left[1 + 2\epsilon\right] \geq 1 \quad \Rightarrow \frac{v_{\perp 0}}{v} \geq 1 - \epsilon \quad \text{For trapped particles}$$



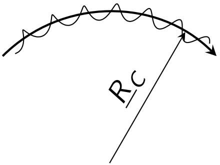
$$\frac{v^2}{v_{\perp 0}^2} \leq 1+2\epsilon \quad \Rightarrow \frac{v_{\perp 0}^2 + v_{||0}^2}{v_{\perp 0}^2} \leq 1+2\epsilon$$

$$\boxed{\frac{v_{||0}}{v_{\perp 0}} \leq \sqrt{2\epsilon}}$$

As particles move around the torus, their orbits drift. We have already come across the ∇B drift:

$$\underline{v}_{\nabla B} = \frac{v_\perp^2}{2\Omega}\frac{\underline{B}\times\nabla\underline{B}}{B^2}$$

but there is also the curvature drift

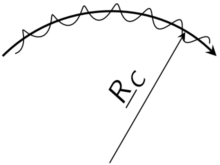


Particle drifts

As particles move around the torus, their orbits drift. We have already come across the ∇B drift:

$$\underline{v}_{\nabla B} = \frac{v_\perp^2}{2\Omega}\frac{\underline{B}\times\nabla\underline{B}}{B^2}$$

but there is also the curvature drift



In the frame of the particle, there is a centrifugal force

$$F_R = \frac{mv_{||}^2\underline{R}_C}{R_C^2}$$

This therefore causes a drift

$$\underline{v}_R = \frac{1}{q}\frac{\left(mv_{||}^2\underline{R}_C/R_C^2\right)\times\underline{B}}{B^2}$$

$$= \frac{m}{qB}\frac{v_{||}^2\underline{R}_C\times\underline{B}}{R_C^2 B} = \frac{v_{||}^2}{\Omega}\frac{\underline{R}_C\times\underline{B}}{R_C^2 B}$$

Particle drifts

So how big are these drifts, and what direction are they in?

$$\underline{v}_{\nabla B} = \frac{v_\perp^2}{2\Omega}\frac{\underline{B}\times\nabla\underline{B}}{B^2}$$

$$\underline{B} \simeq B_0\underline{e}_\phi \quad B_0 \propto \frac{1}{R} \Rightarrow \nabla B \simeq -\frac{B_0}{R}\nabla R$$

$$\frac{\underline{B}\times\nabla\underline{B}}{B^2} \simeq \frac{-\underline{e}_\phi \times \nabla R}{R} = \frac{\underline{e}_Z}{R}$$

$$\Rightarrow \underline{v}_{\nabla B} = \frac{v_\perp^2}{2\Omega R}\underline{e}_Z$$

Particle drifts

So how big are these drifts, and what direction are they in?

$$\underline{v}_{\nabla B} = \frac{v_\perp^2}{2\Omega}\frac{\underline{B}\times\nabla\underline{B}}{B^2}$$

$$\underline{B} \simeq B_0\underline{e}_\phi \quad B_0 \propto \frac{1}{R} \Rightarrow \nabla B \simeq -\frac{B_0}{R}\nabla R$$

$$\frac{\underline{B}\times\nabla\underline{B}}{B^2} \simeq \frac{-\underline{e}_\phi \times \nabla R}{R} = \frac{\underline{e}_Z}{R}$$

$$\Rightarrow \underline{v}_{\nabla B} = \frac{v_\perp^2}{2\Omega R}\underline{e}_Z$$

$$\underline{v}_R = \frac{v_{||}^2}{\Omega}\frac{\underline{R}_C\times\underline{B}}{R_C^2 B}$$

$$\underline{R}_C = R\nabla R$$

$$\begin{aligned}\underline{v}_R &= \frac{v_{||}^2}{\Omega}\frac{\nabla R\times\underline{e}_\phi}{R}\\ &= \frac{v_{||}^2}{\Omega}\frac{\underline{e}_Z}{R}\end{aligned}$$

So how big are these drifts, and what direction are they in?

$$\underline{v}_{\nabla B} = \frac{v_\perp^2}{2\Omega}\frac{\underline{B}\times\nabla\underline{B}}{B^2}$$

$$\underline{B} \simeq B_0\underline{e}_\phi \quad B_0 \propto \frac{1}{R} \Rightarrow \nabla B \simeq -\frac{B_0}{R}\nabla R$$

$$\frac{\underline{B}\times\nabla\underline{B}}{B^2} \simeq \frac{-\underline{e}_\phi \times \nabla R}{R} = \frac{\underline{e}_Z}{R}$$

$$\Rightarrow \underline{v}_{\nabla B} = \frac{v_\perp^2}{2\Omega R}\underline{e}_Z$$

$$\underline{v}_R = \frac{v_{||}^2}{\Omega}\frac{\underline{R}_C\times\underline{B}}{R_C^2 B}$$

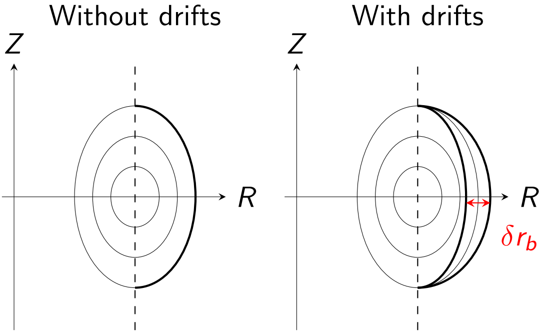
$$\underline{R}_C = R\nabla R$$

$$\underline{v}_R = \frac{v_{||}^2}{\Omega}\frac{\nabla R\times\underline{e}_\phi}{R} = \frac{v_{||}^2}{\Omega}\frac{\underline{e}_Z}{R}$$

Total drift is therefore

$$\underline{v}_{\nabla B} + \underline{v}_R = \frac{\left(v_{||}^2 + v_\perp^2/2\right)}{R\Omega}\underline{e}_Z$$

Note that the drift is in the vertical direction and opposite for electrons and ions



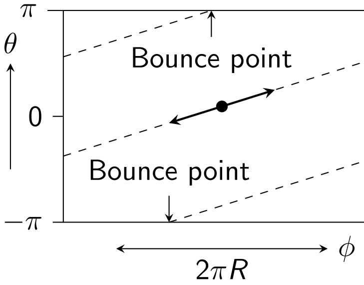
Characteristic shape of the orbits in the poloidal plane gives this the name **banana orbit**. The width of this orbit is the banana width, often denoted δr_{bj} or ρ_{bj} with j indicating electrons or ions.

Banana orbit characteristics

Let us calculate the banana width for a **barely trapped** particle i.e. one with a bounce point at $\theta = \pi$, on the inboard side

Time for half an orbit: Velocity along the field-line $v_{||} \sim \sqrt{2\epsilon} v_\perp$ is small. Total speed is therefore approximately $v \sim v_\perp$. This will be approximately the thermal speed $v \sim v_{th}$.

$$\Rightarrow v_{||} \sim \sqrt{2\epsilon} v_{th} \quad \text{For trapped particles}$$



Distance travelled $= 2\pi R q$ so time for a trapped particle to execute half an orbit

$$t_b = \frac{2\pi R q}{v_{th}\sqrt{2\epsilon}}$$

Banana orbit characteristics

During this time $t_b = 2\pi Rq / \left(v_{th}\sqrt{2\epsilon}\right)$, the particle drifts to a new flux surface, a distance δr_b from the original one.

$$\delta r_b \quad = \quad \left(V_{\nabla B} + V_R\right) \frac{2\pi Rq}{v_{th}\sqrt{2\epsilon}}$$

Banana orbit characteristics

During this time $t_b = 2\pi Rq/\left(v_{th}\sqrt{2\epsilon}\right)$, the particle drifts to a new flux surface, a distance δr_b from the original one.

$$\delta r_b \;=\; \left(V_{\nabla B} + V_R\right)\frac{2\pi Rq}{v_{th}\sqrt{2\epsilon}} \simeq \frac{1}{R\Omega}\left[\underbrace{2\epsilon v_{th}^2}_{v_{||}^2} + \underbrace{\frac{v_{th}^2}{2}}_{v_\perp^2/2}\right]\frac{2\pi Rq}{v_{th}\sqrt{2\epsilon}}$$

Banana orbit characteristics

During this time $t_b = 2\pi Rq/\left(v_{th}\sqrt{2\epsilon}\right)$, the particle drifts to a new flux surface, a distance δr_b from the original one.

$$\begin{aligned}\delta r_b &= \left(V_{\nabla B} + V_R\right)\frac{2\pi Rq}{v_{th}\sqrt{2\epsilon}} \simeq \frac{1}{R\Omega}\left[\,\underbrace{2\epsilon v_{th}^2}_{v_{||}^2} + \underbrace{\frac{v_{th}^2}{2}}_{v_\perp^2/2}\,\right]\frac{2\pi Rq}{v_{th}\sqrt{2\epsilon}} \\ &= \frac{\pi}{\sqrt{2}}\underbrace{\frac{v_{th}}{\Omega}}_{r_L}\frac{\left(4\epsilon + 1\right)}{\sqrt{\epsilon}}q \simeq \frac{\pi}{\sqrt{2}}\frac{r_L q}{\sqrt{\epsilon}}\end{aligned}$$

Note that this is much larger than the Larmor radius r_L
$\Rightarrow$ does this provide another transport mechanism?

Collisions

We've already seen collisions, and come across the collision times

$$\tau_{ei} < \tau_{ii} \sim \sqrt{\frac{m_i}{m_e}}\frac{1}{Z^2} < \tau_{ie} \sim \frac{m_i}{m_e}\tau_{ei}$$

τ_{jk} average times it takes to change the velocity of particles of species j by 90^o, through scattering with species k.

- To take a step of size δr_b, a particle doesn't need to be deflected by 90^o. It just needs to be scattered from a trapped into a passing particle.
- To do this, the parallel velocity needs to be changed by $\Delta v_{||} \sim \sqrt{\epsilon} v_{th}$
- The **effective collision time** is therefore $\tau_{eff} \sim \tau\epsilon$

We can also define a collision frequency, which is just $\boxed{\nu \equiv \frac{1}{\tau}}$, and so an effective collision frequency for trapped particles

$$\nu_{eff} \sim \frac{\nu}{\epsilon} = \frac{1}{\tau\epsilon}$$

Collisionality

A useful quantity used in MCF is the **collisionality** ν_*. This is the average number of times a particle is scattered into a passing particle before completing a banana orbit.

$$\boxed{\nu_* \equiv \frac{t_b}{\tau_{eff}}}$$

Collisionality

A useful quantity used in MCF is the **collisionality** ν_*. This is the average number of times a particle is scattered into a passing particle before completing a banana orbit.

$$\boxed{\nu_* \equiv \frac{t_b}{\tau_{eff}}} = \frac{\nu}{\epsilon}\frac{Rq}{\sqrt{\epsilon}v_{th}}$$

Collisionality

A useful quantity used in MCF is the **collisionality** ν_*. This is the average number of times a particle is scattered into a passing particle before completing a banana orbit.

$$\boxed{\nu_* \equiv \frac{t_b}{\tau_{eff}}} = \frac{\nu}{\epsilon}\frac{Rq}{\sqrt{\epsilon}v_{th}}$$

$$\nu_* = \frac{\nu Rq}{\epsilon^{3/2}v_{th}}$$

Note that if $\nu_* > 1$ then trapped particles do not complete a full banana orbit before being scattered. Thus trapped particles only exist for $\nu_* < 1$.

Collisionality

For electrons colliding with ions or electrons,

$$\tau_e \propto \frac{m_e^{1/2} T_e^{3/2}}{n} \Rightarrow \nu_{*e} \propto \frac{nRq}{\epsilon^{3/2} T_e^2}$$

For ions colliding with ions (ion-electron negligible)

$$\tau_i \propto \frac{m_i^{1/2} T_e^{3/2}}{n} \Rightarrow \nu_{*i} \propto \frac{nRq}{\epsilon^{3/2} T_i^2}$$

Collisionality

For electrons colliding with ions or electrons,

$$\tau_e \propto \frac{m_e^{1/2} T_e^{3/2}}{n} \Rightarrow \nu_{*e} \propto \frac{nRq}{\epsilon^{3/2} T_e^2}$$

For ions colliding with ions (ion-electron negligible)

$$\tau_i \propto \frac{m_i^{1/2} T_e^{3/2}}{n} \Rightarrow \nu_{*i} \propto \frac{nRq}{\epsilon^{3/2} T_i^2}$$

Note that

- The ratio $\nu_i / \nu_e \sim \sqrt{m_e / m_i} \ll 1$
- Collisionality is independent of mass ($\sim$ equal for ions and electrons)
- $\nu_* \propto n / T^2 \Rightarrow$ very low for hot tokamaks so trapped particles become more important

Neoclassical transport

At low collisionality When $\nu_* \ll 1$, trapped particles exist for many banana orbits. This is called the **banana regime**.

After N steps in a random direction, particles or energy will diffuse an average of $\sqrt{N}$ steps

$$N \sim \frac{t}{\tau_{eff}}\sqrt{2\epsilon}$$

Note that N is multiplied by the fraction of trapped particles.

Neoclassical transport

At low collisionality When $\nu_* \ll 1$, trapped particles exist for many banana orbits. This is called the **banana regime**.

After N steps in a random direction, particles or energy will diffuse an average of $\sqrt{N}$ steps

$$N \sim \frac{t}{\tau_{eff}}\sqrt{2\epsilon}$$

Note that N is multiplied by the fraction of trapped particles.

The typical distance energy diffuses in a time t is

$$d_{neo} \sim \sqrt{\frac{t\sqrt{2\epsilon}}{\tau_{eff}}}\delta r_b = \sqrt{\frac{t}{\tau}\sqrt{\frac{2}{\epsilon}}}\delta r_b \simeq \underbrace{\frac{\pi}{2^{1/4}}\frac{q}{\epsilon^{3/4}}}_{\sim 10-30} \underbrace{\sqrt{\frac{t}{\tau_{ii}}}r_{Li}}_{\text{Classical result}}$$

- Classical transport $\rightarrow$ needed minor radius $r \sim 14$cm
- Neoclassical transport increases this by ~ 10.
- ITER (expected $Q = 10$) has a minor radius of ~ 2m.
- Most transport in tokamaks is **anomalous**, due to turbulence

Diffusion equations

Consider a volume of plasma containing a particle density n and energy density nT

Flux of particles is $\underline{\Gamma}$

Flux of energy is $\underline{Q}$

Diffusion equations

Consider a volume of plasma containing a particle density n and energy density nT

Flux of particles is $\underline{\Gamma}$

Flux of energy is $\underline{Q}$

Continuity gives

$$\frac{\partial n}{\partial t} = -\nabla \cdot \underline{\Gamma}$$

$$\frac{\partial}{\partial t}\left(nT\right) = -\nabla \cdot \underline{Q}$$

Diffusion equations

Consider a volume of plasma containing a particle density n and energy density nT

Flux of particles is $\underline{\Gamma}$

Flux of energy is $\underline{Q}$

Continuity gives

$$\frac{\partial n}{\partial t} = -\nabla \cdot \underline{\Gamma}$$

$$\frac{\partial}{\partial t}(nT) = -\nabla \cdot \underline{Q}$$

Diffusive process: flux $\propto$ gradient (Fick's law)

$$\boxed{\underline{\Gamma} = -D\nabla n}$$

$$\boxed{\underline{Q} = -n\chi\nabla T}$$

$$\frac{\partial n}{\partial t} = \nabla \cdot (D\nabla n)$$

so if D is approximately constant:

$$\frac{\partial n}{\partial t} = D\nabla^2 n$$

$$\frac{\partial}{\partial t}(nT) = \nabla \cdot (n\chi\nabla T)$$

Assuming n constant:

$$\frac{\partial T}{\partial t} = \nabla \cdot (\chi\nabla T)$$

Diffusion equations

- From the units of D and χ (L^2/T), they must be the step size squared over the step time.
- For classical transport,

$$D_i = D_e \sim r_{Le}^2/\tau_{ei} \simeq 3 \times 10^{-4} m^2/s$$

$$\chi_i \sim r_{Li}^2/\tau_{ii} \simeq 2 \times 10^{-2} m^2/s \quad \chi_e \sim r_{Le}^2/\tau_{ei} \simeq 3 \times 10^{-4} m^2/s$$

- From the units of D and χ (L^2/T), they must be the step size squared over the step time.
- For classical transport,

$$D_i = D_e \sim r_{Le}^2/\tau_{ei} \simeq 3 \times 10^{-4} m^2/s$$

$$\chi_i \sim r_{Li}^2/\tau_{ii} \simeq 2 \times 10^{-2} m^2/s \quad \chi_e \sim r_{Le}^2/\tau_{ei} \simeq 3 \times 10^{-4} m^2/s$$

- For neoclassical transport,

$$\chi_i \sim \sqrt{2\epsilon}\delta r_{bi}^2/\tau_{eff} \sim 0.4 m^2/s$$

$$\chi_e \sim \chi_i \underbrace{\frac{\delta r_{be}^2}{\delta r_{bi}^2}}_{\sim m_e/m_i} \times \underbrace{\frac{\tau_{ii}}{\tau_{ei}}}_{\sim\sqrt{m_i/m_e}} = \chi_i\sqrt{\frac{m_e}{m_i}} \simeq \chi_i/60 \sim 7\times10^{-3} m^2/s$$

What about neoclassical particle transport $D_{i,e}$?

Neoclassical particle transport

- For classical transport, collisions between particles of the same species didn't contribute to particle transport
- In this case most of the particles a trapped particle is colliding with are passing particles so this is no longer true

$\chi_i \sim 60\chi_e$ so does this mean that $D_i \sim 60D_e$?

Neoclassical particle transport

- For classical transport, collisions between particles of the same species didn't contribute to particle transport
- In this case most of the particles a trapped particle is colliding with are passing particles so this is no longer true

$$\chi_i \sim 60\chi_e \text{ so does this mean that } D_i \sim 60 D_e \text{ ?}$$

- If this happened (and it does in some situations), the plasma would start to charge up, creating an electric field which held the ions back: this is called **non-ambipolar** transport
- It turns out that due to momentum conservation, ions and electrons actually diffuse at the same rate without an electric field (**intrinsically ambipolar**)
- Neoclassical particle diffusivity is comparable to χ_e

$$\chi_e \sim D_e \sim D_i \sim \frac{q^2}{\epsilon^{3/2}} r_{Le}^2/\tau_{ei} \quad \chi_i \sim \sqrt{\frac{m_i}{m_e}}\chi_e$$

Summary

- In toroidal machines, the variation in magnetic field strength leads to particle trapping
- The Grad-B and curvature drifts cause trapped particles to follow **banana orbits**
- Collisions scatter trapped particles into passing particles, and **collisionality** ν_* is the average number effective collision times $\tau_{eff} \sim \tau\epsilon$ per banana orbit
- This leads to a diffusion with a step size of the banana width δr_b and time scale τ_{eff}: $\chi \sim \delta r_b^2/\tau_{eff}$
- This **neoclassical** transport is the minimum possible in a toroidal device
- Measured diffusivities in tokamaks are typically $\sim 10 - 100$ times larger than neoclassical: they are **anomalous**
- This is due to turbulence, which we'll study later in the course