MHD equilibrium

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Magnetic mirror effect leads to particle trapping in toroidal machines.

Trapped particles have “banana” orbits which lead to neoclassical transport.

This gives the minimum possible transport in a given configuration.

How is this magnetic configuration determined?
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In a tokamak, this field is produced in a large part by the plasma itself. We therefore need to find a self-consistent way to solve for both the magnetic field and the particles.

The particles are in a 6-dimensional phase space $f(x, v)$

(Note: even getting to here involves approximations)

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial x} + q (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = C(f)$$

plus the equations of electromagnetism (minus displacement current)

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \quad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$
Unfortunately, solving this is impossible for most problems.

- This equation can be solved for simple systems (e.g. 1D)

\[
\langle f \rangle_n = \int v^n f \, dv
\]

The zeroth moment \((n = 0)\) is density, first the average velocity, second the energy, third the heat flux, ... Unfortunately, each moment depends on the next one so assumptions need to be made to solve the equations.

By assuming the plasma is adiabatic (which links pressure and density) and adding the equations for electrons and ions we get a particularly useful set of equations: ideal MHD.
Fluid equations

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- By assuming the plasma is adiabatic (which links pressure and density) and adding the equations for electrons and ions we get a particularly useful set of equations: **ideal MHD**
Ideal MagnetoHydroDynamics (MHD) is a set of equations for the mass density $\rho$, velocity $\mathbf{v}$ and pressure $P$ of a conducting fluid.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (1)$$

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla P + \mathbf{J} \times \mathbf{B} \quad (2)$$

$$\frac{\partial P}{\partial t} + \mathbf{v} \cdot \nabla P = -\gamma P \nabla \cdot \mathbf{v} \quad (3)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) \quad (4)$$
The main assumptions made in deriving ideal MHD are

- Quasineutrality $n_i \approx n_e$
- Length scales $\gg$ Larmor radius
- Frequencies $\ll$ Cyclotron frequency
- No electron inertia ($m_e = 0$)
- Hall current neglected (no $j \times B$ term in Ohm’s law)
- High collision rate (so nearly maxwellian)

- No dissipation: zero viscosity and resistivity
- No trapped particles, so no neoclassical effects
MHD equilibrium

- Find plasma configurations which are in equilibrium
  \[ \frac{\partial}{\partial t} = 0 \]
- To simplify things, also assume a stationary equilibrium so \( v = 0 \). This is reasonable if flow velocities are much less than the sound speed

This leaves us with just one expression which must be satisfied:

\[ J \times B = \nabla P \]
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This leaves us with just one expression which must be satisfied:
\[ \vec{J} \times \vec{B} = \nabla P \]

This equation has a few implications:
- \[ \vec{B} \cdot \nabla P = \vec{B} \cdot (\vec{J} \times \vec{B}) = 0 \]
  Magnetic field \( \perp \) to pressure gradient
- \[ \vec{J} \cdot \nabla P = \vec{J} \cdot (\vec{J} \times \vec{B}) = 0 \]
  Current also \( \perp \) to the pressure gradient
- We want a configuration in which the pressure is high in the middle of the plasma, and low at the edge.
- Imagine a contour plot of the pressure:
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Imagine a contour plot of the pressure:

\[ B \cdot \nabla P = 0 \] says that \( B \) must also lie on these surfaces.

These are of course the flux surfaces we saw in lecture 2.
How does the magnetic field balance the plasma pressure? Use Ampére’s law to eliminate $J$

$$\frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla P$$

The following vector identity

$$\mathbf{B} \times (\nabla \times \mathbf{B}) = \frac{1}{2} \nabla B^2 - (\mathbf{B} \cdot \nabla) \mathbf{B}$$

then gives

$$\nabla \left( P + \frac{B^2}{2\mu_0} \right) - \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} = 0$$
After a bit of manipulation, this can be re-written as

$$\nabla_{\perp} \left( P + \frac{B^2}{2\mu_0} \right) - \frac{B^2}{\mu_0} (\mathbf{b} \cdot \nabla) \mathbf{b} = 0$$

where $\mathbf{b} = B/B$ is the unit $B$ vector, and $\nabla_{\perp} = \nabla - \mathbf{b} (\mathbf{b} \cdot \nabla)$ is the gradient perpendicular to $\mathbf{B}$. 
After a bit of manipulation, this can be re-written as

\[ \nabla_\perp \left( P + \frac{B^2}{2\mu_0} \right) - \frac{B_s^2}{\mu_0} (b \cdot \nabla) b = 0 \]

where \( \underline{b} = \underline{B}/B \) is the unit \( \underline{B} \) vector, and \( \nabla_\perp = \nabla - \underline{b} (\underline{b} \cdot \nabla) \) is the gradient perpendicular to \( \underline{B} \). This last term is the gradient of \( \underline{b} \) in the direction of \( \underline{b} \). This is the curvature

\[ \kappa \equiv (b \cdot \nabla) b = -\frac{R_C}{R_C^2} \]

where \( R_C \) is the radius of curvature
Magnetic pressure and tension

\[ \nabla \perp \left( P + \frac{B^2}{2\mu_0} \right) - \frac{B^2}{\mu_0} (b \cdot \nabla) b = 0 \]

- Magnetic fields exert a pressure on the plasma

Figure: Θ-pinch. J. Friedberg, Plasma Physics and Fusion Energy
Magnetic pressure and tension

\[ \nabla_\perp \left( P + \frac{B^2}{2\mu_0} \right) - \frac{B^2}{\mu_0} (b \cdot \nabla) b = 0 \]

- Magnetic fields exert a pressure on the plasma.
- They also have a tension which tries to straighten them.

**Figure:** Z-pinch. J.Friedberg, Plasma Physics and Fusion Energy
Calculating tokamak equilibria

How do we calculate a solution to \( J \times B = \nabla P \) in a tokamak?

- First we need to introduce a way to label our flux surfaces

\[ \psi = \int B \cdot dS \]

\( \Rightarrow \psi \) is a flux-surface label
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Integrate the **magnetic flux** $\psi = \int B \cdot dS$

As $B$ lies on flux-surfaces, it doesn’t matter which path our cut takes. $\Rightarrow \psi$ is a flux-surface label.
Using $\psi$, we can write the poloidal field in a nice way:

$$d\psi = B \cdot dS = B_\theta \cdot Rdl \Rightarrow \nabla \psi \propto B_\theta R$$

**Note:** Area $dS$ is $2\pi R \times dl$ but the $2\pi$ is usually dropped from definition of $\psi$

The poloidal magnetic field is perpendicular to $\nabla \psi$ and $e_\phi$ and can be written as

$$B_\theta = \frac{1}{R} \nabla \psi \times e_\phi = \nabla \psi \times \nabla \phi$$
The toroidal magnetic field has two sources:

- external coils producing the vacuum field
- the poloidal current induced in the plasma which partly confines the plasma and acts to reduce the toroidal field

For a tokamak then it will be some function of $R$ and $Z$:

$$B_\phi = f(R, Z) \nabla \phi$$

Using Ampère’s law to get the poloidal current gives:

$$\mu_0 j_\theta = \nabla \times (f(R, Z) \nabla \phi) = -\nabla \phi \times \nabla f(R, Z)$$

Since $j_\theta \cdot \nabla \psi = 0$, we get

$$\nabla \psi \cdot (-\nabla \phi \times \nabla f(R, Z)) = -\nabla \phi \cdot (\nabla f(R, Z) \times \nabla \psi) = 0$$

Hence $\nabla f(R, Z) \times \nabla \psi = 0$. The only way this can be true is if $f$ is constant on flux surface i.e. $f = f(\psi)$
Combining the poloidal and toroidal components, we can write the total magnetic field as

\[ \mathbf{B} = f(\psi) \nabla \phi + \nabla \psi \times \nabla \phi \]

This means that to describe our magnetic field everywhere we just need \( \psi (R, Z) \) and \( f(\psi) = RB_\phi \).
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Taking the \(\nabla \psi\) component of the force balance \(\mathbf{J} \times \mathbf{B} = \nabla p\) and some manipulation gives the Grad-Shafranov equation:

$$R \frac{\partial}{\partial R} \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial \psi}{\partial Z} = -\mu_0 R^2 \frac{\partial p(\psi)}{\partial \psi} - \mu_0^2 f(\psi) \frac{\partial f(\psi)}{\partial \psi}$$

This is a nonlinear PDE involving \(\psi(R, Z)\), \(f(\psi)\) and \(p(\psi)\), and is used to design and interpret tokamak experiments.
Solving the Grad-Shafranov equation

- This equation is a non-linear partial differential equation, and in general can’t be solved analytically.
- Instead, we need to find solutions numerically. Two main types of code:
  - Forward codes, which calculate an equilibrium from \( p(\psi) \) and \( f(\psi) \) (or \( q(\psi) \) or \( j_{||}(\psi) \)) and either the plasma boundary shape or coil currents.
  - **Examples**: SCENE, CORSICA, TEQ
- Primarily used by theorists or tokamak designers.
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- **Interpretive codes**, which take experimental measurements and work out the equilibrium.
  
  **Examples**: EFIT, CLISTE
  
  ⇒ Used by experimentalists to analyse experimental data.
Typical MAST results

A typical contour plot of $\psi$ from MAST (calculated using EFIT) looks something like:

- Contours of $\psi$ in black
- Simplified boundary in red
- Dense contours towards the right side are the poloidal field coils for plasma shaping and vertical stability
- Outside the core, plasma has a double null x-point configuration
X-point equilibria

- Poloidal field due to the plasma current
X-point equilibria

- Poloidal field due to the plasma current
- Add another coil carrying a current in the same direction
X-point equilibria

- Poloidal field due to the plasma current
- Add another coil carrying a current in the same direction
- At some point the poloidal field cancels, and forms an x-point
- The field line through the x-point is called a **separatrix**
All large tokamaks use additional coils to produce one or two x-points where the poloidal field cancels out. There are several reasons for this:

- Plasma which escapes the core is channelled along the divertor legs to specially armored regions which can handle the high heat load.
- Separating the plasma from the surfaces reduces contamination of the plasma.
- For reasons which aren’t clear, turbulence can be suppressed near the edge of x-point configurations (H-mode).
One common parameterisation of plasma shape is\textsuperscript{a}:

\begin{align*}
R &= R_0 - b + (a + b \cos \theta) \cos (\theta + \delta \sin \theta) \\
Z &= \kappa a \sin \theta
\end{align*}

- Major radius $R_0$
- Minor radius $a$
- Ellipticity $\kappa$, which is 1 for a circle.

Often see \textbf{elongation} $= \text{ellipticity} - 1$ so zero for a circle

\textsuperscript{a}J. Manickam, Nucl. Fusion \textbf{24} 595 (1984)
Plasma shaping

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- **Triangularity** \( \delta \) (‘D’ shape)

\( a = 1, \kappa = 1.4, \delta = 0.5 \)

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- **Minor radius** $a$
- **Ellipticity** $\kappa$, which is 1 for a circle.
- Often see **elongation** = ellipticity - 1 so zero for a circle
- **Triangularity** $\delta$ (’D’ shape)
- **Indentation** $b$ (’bean’ shape)

Ideal MHD is a simplified description of a plasma which describes many phenomena in plasmas.

Equilibrium is given by solutions to $J \times B = \nabla P$.

In axisymmetric configurations with flux surfaces, this can be simplified to the **Grad-Shafranov equation**.

Realistic configurations have x-points where the poloidal field goes to zero (but the toroidal field is not zero).

Poloidal field coils are used to shape the plasma into configurations characterised by **elongation** and **triangularity**. This affects plasma performance and stability.