MHD waves

Ben Dudson

Department of Physics, University of York
Heslington, York YO10 5DD, UK

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Last time

- Last time: Ohmic heating and current drive, and it's limits
- Neutral beams used to supply heating power and drive current
- Many other methods for plasma heating and current drive involve exciting waves in the plasma
- To study these, we'll need to look at plasma waves, and resonant frequencies
- Alfvèn waves in confined plasmas are a rich area of study, and their interaction with fast particles such as Neutral Beam particles and fusion alphas is important for reactors. Many acronyms refer to these: TAEs, CAEs, EAEs, NAEs, ...
- Later in the course we'll use similar methods to study instabilities

MHD

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\underline{v}\cdot\nabla\rho - \boxed{\rho\nabla\cdot\underline{v}} \\
\frac{\partial \underline{v}}{\partial t} &= -\underline{v}\cdot\nabla\underline{v} - \frac{1}{\rho}\nabla P + \frac{1}{\mu_0\rho}\left(\nabla\times\underline{B}\right)\times\underline{B} \\
\frac{\partial P}{\partial t} &= -\underline{v}\cdot\nabla P - \boxed{\gamma P\nabla\cdot\underline{v}} \\
\frac{\partial \underline{B}}{\partial t} &= -\underline{v}\cdot\nabla\underline{B} + \underline{B}\cdot\nabla\underline{v} - \boxed{\underline{B}\nabla\cdot\underline{v}} - \boxed{\eta\nabla\times\underline{J}}
\end{aligned}
$$

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- Compressional terms

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- Compressional terms and Resistive term

MHD

$$\frac{\partial \rho}{\partial t} = -\underline{v}\cdot\nabla\rho - \rho\nabla\cdot\underline{v}$$

$$\frac{\partial \underline{v}}{\partial t} = -\underline{v}\cdot\nabla\underline{v} - \frac{1}{\rho}\nabla P + \frac{1}{\mu_0\rho}\left(\nabla\times\underline{B}\right)\times\underline{B}$$

$$\frac{\partial P}{\partial t} = -\underline{v}\cdot\nabla P - \gamma P\nabla\cdot\underline{v}$$

$$\frac{\partial \underline{B}}{\partial t} = -\underline{v}\cdot\nabla\underline{B} + \underline{B}\cdot\nabla\underline{v} - \underline{B}\nabla\cdot\underline{v} - \eta\nabla\times\underline{J}$$

- Compressional terms and Resistive term
- This involves 8 fields which we can write as a state vector

$$\underline{X} = \left(\rho, \underline{v}, P, \underline{B}\right)$$

- The equations of MHD can therefore be written as

$$\frac{\partial \underline{X}}{\partial t} = \underline{F}\left(\underline{X}\right)$$

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For an equilibrium $\underline{X}_0 = \left(\rho_0, \underline{v}_0, P_0, \underline{B}_0\right)$, the time derivatives are zero

$$\frac{\partial \underline{X}_0}{\partial t} = \underline{F}\left(\underline{X}_0\right) = 0$$

Having found an equilibrium, we can give it a small perturbation
This can lead to either oscillating waves or growing instabilities
This is done by expanding in a small parameter ϵ

$$\underline{X} = \underline{X}_0 + \epsilon \underline{X}_1 + \epsilon^2 \underline{X}_2 + \dots$$

This is called **linearisation**, because it allows us to eliminate nonlinear terms and get a set of equations which we can attack using linear algebra.

Linearisation

For example the density equation:

$$\frac{\partial \rho}{\partial t} = -\underline{v} \cdot \nabla \rho - \rho \nabla \cdot \underline{v}$$

we expand $\rho = \rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \ldots$ and $\underline{v} = \underline{v}_0 + \epsilon\underline{v}_1 + \epsilon^2\underline{v}_2 + \ldots$

Linearisation

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$$\begin{aligned} \frac{\partial \rho_0}{\partial t} + \epsilon \frac{\partial \rho_1}{\partial t} + \epsilon^2 \frac{\partial \rho_2}{\partial t} + \dots = \\ -\left(\underline{v}_0 + \epsilon \underline{v}_1 + \epsilon^2 \underline{v}_2\right) \cdot \nabla \left(\rho_0 + \epsilon \rho_1 + \epsilon^2 \rho_2\right) + \dots \\ -\left(\rho_0 + \epsilon \rho_1 + \epsilon^2 \rho_2\right) \nabla \cdot \left(\underline{v}_0 + \epsilon \underline{v}_1 + \epsilon^2 \underline{v}_2\right) + \dots \end{aligned}$$

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Group these terms into powers of ϵ

$$\left[\frac{\partial \rho_0}{\partial t} + \underline{v}_0 \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \underline{v}_0\right] +$$

$$\epsilon \left[\frac{\partial \rho_1}{\partial t} + \underline{v}_1 \cdot \nabla \rho_0 + \underline{v}_0 \cdot \nabla \rho_1 + \rho_1 \nabla \cdot \underline{v}_0 + \rho_0 \nabla \cdot \underline{v}_1\right] + \epsilon^2 \left[\cdot\right] + \ldots = 0$$

ϵ is an arbitrary small parameter $\Rightarrow$ each bracket must be zero

Linearisation

The first bracket is of course just the equilibrium

$$\frac{\partial \rho_0}{\partial t} + \underline{v}_0 \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \underline{v}_0 = 0$$

The second bracket ($\propto \epsilon$)

$$\frac{\partial \rho_1}{\partial t} + \underline{v}_1 \cdot \nabla \rho_0 + \underline{v}_0 \cdot \nabla \rho_1 + \rho_1 \nabla \cdot \underline{v}_0 + \rho_0 \nabla \cdot \underline{v}_1 = 0$$

This is a linear equation in ρ_1 and $\underline{v}_1$, and represents what happens if an equilibrium is given a small perturbation

Solving linear MHD

The linear equations we get from this are still a set of coupled partial differential equations, so still quite hard to solve. Some approaches to this are:

1. Use a numerical grid (discrete points or basis functions), and discretise numerically. This turns linear PDEs into linear algebraic equations

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2. Simplify the equations using e.g. $k_{||} \ll k_{\perp}$ to separate parallel and perpendicular directions. Multi-scale perturbation methods can sometimes be applied to find solutions e.g. WKB

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2. Simplify the equations using e.g. $k_{||} \ll k_\perp$ to separate parallel and perpendicular directions. Multi-scale perturbation methods can sometimes be applied to find solutions e.g. WKB
3. Assume **homogenous** plasma, so that we can take Fourier transforms in each direction. All quantities written in the form

$$f\left(\underline{x}, t\right) = \int\int \hat{f}\left(\underline{k}, \omega\right) e^{i\left(\underline{k}\cdot\underline{x} - \omega t\right)} d\underline{k} d\omega$$

This allows us to replace

$$\nabla \rightarrow i\underline{k} \qquad \frac{\partial}{\partial t} \rightarrow -i\omega$$

Homogeneous approximation

If we make the further assumption that the equilibrium is stationary, so $\underline{v}_0 = 0$ then the linear equation

$$\frac{\partial \rho_1}{\partial t} + \underline{v}_1 \cdot \nabla \rho_0 + \underline{v}_0 \cdot \nabla \rho_1 + \rho_1 \nabla \cdot \underline{v}_0 + \rho_0 \nabla \cdot \underline{v}_1 = 0$$

becomes

$$-i\omega\rho_1 = -\rho_0 i\underline{k} \cdot \underline{v}_1$$

because all derivatives of equilibrium quantities are zero (homogenous).
note: Hats are often dropped from e.g. $\hat{\rho}_1$ when it's clear which one is meant.

Homogeneous approximation

The equations of ideal MHD ($\eta = 0$) can then be written

$$
\begin{aligned}
-i\omega\rho_1 &= -\rho_0 i\underline{k}\cdot\underline{v}_1 \\
-i\omega\underline{v}_1 &= -\frac{p_1}{\rho_0}i\underline{k} + \frac{1}{\mu_0\rho_0}\left(i\underline{k}\times\underline{B}_1\right)\times\underline{B}_0 \\
-i\omega p_1 &= -\gamma p_0 i\underline{k}\cdot\underline{v}_1 \\
-i\omega\underline{B}_1 &= \left(i\underline{k}\cdot\underline{B}_0\right)\underline{v}_1 - \left(i\underline{k}\cdot\underline{v}_1\right)\underline{B}_0
\end{aligned}
$$

We've now got rid of all the differential operators, and are left with a set of (complex) coupled linear equations.
This set of equations have some nice properties:

1. It's Hermitian (complex conjugate equals transpose), so ω^2 is real. This means waves are either oscillating ($\omega^2 > 0$) or growing/shrinking ($\omega^2 < 0$), but never both.

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This set of equations have some nice properties:

1. It's Hermitian (complex conjugate equals transpose), so ω^2 is real. This means waves are either oscillating ($\omega^2 > 0$) or growing/shrinking ($\omega^2 < 0$), but never both.
2. We can eliminate almost all these variables, and reduce this to an equation for $\underline{v}$ only

Homogeneous approximation

Without loss of generality, we can choose $\underline{B}$ to be along the z axis, and $\underline{k}$ to be in the $x - z$ plane:

$$\underline{B}_0 = B_0 \underline{e}_z \qquad \underline{k} = k_x \underline{e}_x + k_z \underline{e}_z$$

Writing out the components of the equation for $\underline{v}$ gives:

$$\left(\begin{array}{ccc} \omega^2 - k^2 V_A^2 - k^2 V_s^2 \sin^2\theta & 0 & -k^2 V_s^2 \sin\theta\cos\theta \\ 0 & \omega^2 - k^2 V_A^2 \cos^2\theta & 0 \\ -k^2 V_s^2 \sin\theta\cos\theta & 0 & \omega^2 - k^2 V_s^2 \cos^2\theta \end{array}\right)\left(\begin{array}{c} v_{1x} \\ v_{1y} \\ v_{1z} \end{array}\right) = \underline{0}$$

where θ is the angle between the wave vector $\underline{k}$ and the equilibrium magnetic field $\underline{B}_0$. The Alfvén and sound wave speeds are

$$V_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}} \qquad V_s = \sqrt{\frac{\gamma P_0}{\rho_0}}$$

Wave solutions

To find the solutions to this:

$$\left(\begin{array}{ccc} \omega^2 - k^2V_A^2 - k^2V_s^2\sin^2\theta & 0 & -k^2V_s^2\sin\theta\cos\theta \\ 0 & \omega^2 - k^2V_A^2\cos^2\theta & 0 \\ -k^2V_s^2\sin\theta\cos\theta & 0 & \omega^2 - k^2V_s^2\cos^2\theta \end{array}\right)\left(\begin{array}{c} v_{1x} \\ v_{1y} \\ v_{1z} \end{array}\right) = \underline{0}$$

find where the matrix determinant is zero:

$$\left(\omega^2 - k^2V_A^2\cos^2\theta\right)\left[\omega^4 - \omega^2k^2\left(V_A^2 + V_s^2\right) + k^4V_A^2V_s^2\cos^2\theta\right] = 0$$

This is cubic in ω^2, so has three solutions

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This is cubic in ω^2, so has three solutions

1. **Shear Alfvén wave**: $\omega^2 - k^2 V_A^2 \cos^2\theta = 0$. Since $k\cos\theta = k_{||}$, this gives $\omega^2 = k_{||}^2 V_A^2$
   - Phase speed $\frac{\omega}{k} = \pm V_A$, group speed $\frac{\partial\omega}{\partial k} = \pm V_A$
   - Shear wave: $\underline{v}_1 \perp \underline{B}_0$, $\underline{v}_1 \perp \underline{k}$, $\underline{B}_1 \perp \underline{B}_0$
   - Incompressible: $P_1 = 0$, $\rho_1 = 0$

Wave solutions

2. **Fast magnetosonic wave**:

$$\frac{\omega^2}{k^2} = \frac{1}{2}\left[V_A^2 + V_s^2 + \sqrt{\left(V_A^2 + V_s^2\right)^2 - 4V_A^2V_s^2\cos^2\theta}\right]$$

- Compressible wave: $P_1 \neq 0$, $\rho_1 \neq 0$
- In cold plasma, $V_s = 0$ and $\omega^2 = k^2V_A^2$

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- Compressible wave: $P_1 \neq 0$, $\rho_1 \neq 0$
- In cold plasma, $V_s = 0$ and $\omega^2 = k^2V_A^2$

3. **Slow magnetosonic wave**:

$$\frac{\omega^2}{k^2} = \frac{1}{2}\left[V_A^2 + V_s^2 - \sqrt{\left(V_A^2 + V_s^2\right)^2 - 4V_A^2V_s^2\cos^2\theta}\right]$$

- Compressible wave: $P_1 \neq 0$, $\rho_1 \neq 0$
- For $V_s \ll V_A$, $\omega \simeq kV_s\cos\theta = k_{||}V_s$
- This is a sound wave propagating along $\underline{B}_0$

Alfvénic eigenmodes

On small scales the plasma may be locally homogenous, but on scales comparable to machine size it's definitely not. What do these waves look like?

- Still use small amplitude linearisation
- Single frequency, so $\frac{\partial}{\partial t} \rightarrow -i\omega$
- Can't assume homogenous equilibrium, so $\nabla \neq i\underline{k}$

In a toroidally symmetric equilibrium, we can simplify the ∇ operators slightly by Fourier transforming in toroidal angle ϕ, so $\frac{\partial}{\partial \phi} \rightarrow -in$ where n is the **toroidal mode number**

In the R-Z plane, we then need to solve numerically to obtain solutions. First, we can simplify the geometry to get some understanding

Large aspect-ratio approximation

A useful simplification is to study a large aspect ratio tokamak with circular cross-section, i.e. a cylinder.

In this limit, the plasma is symmetric in both toroidal and poloidal directions so we can Fourier transform, and assume all quantities vary like:

$$f\left(r,\theta,\phi,t\right) = \int\sum_m\sum_n \hat{f}\left(r,m,n,\omega\right)e^{i\left(m\theta - n\phi - \omega t\right)}d\omega$$

(**Note** the sum, since only integer m and n are allowed)
Differential operators can then be simplified e.g.

$$\begin{aligned}\nabla f &= \underline{\hat{r}}\frac{\partial f}{\partial r} + \underline{\hat{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta} + \underline{\hat{\phi}}\frac{1}{R}\frac{\partial f}{\partial \phi} \\ &\rightarrow \underline{\hat{r}}\frac{\partial \hat{f}}{\partial r} + \underline{\hat{\theta}}\frac{im}{r}\hat{f} - \underline{\hat{\phi}}\frac{in}{R}\hat{f}\end{aligned}$$

Large aspect-ratio approximation

This reduces the 3-dimensional PDE to a 1-D ODE in r:

$$\frac{d}{dr}\left[\left(\rho_0\omega^2 - F^2\right) r^3 \frac{d\xi_r}{dr}\right] - \left(m^2 - 1\right)\left[\rho\omega^2 - F^2\right] r\xi_r + \omega^2 r^2 \frac{d\rho_0}{dr}\xi_r = 0$$

where $\underline{\xi}$ is the displacement, so $\underline{v} = \frac{\partial \underline{\xi}}{\partial t}$. The parameter F is

$$F = (m - nq)\frac{B_0}{r\sqrt{\mu_0}}$$

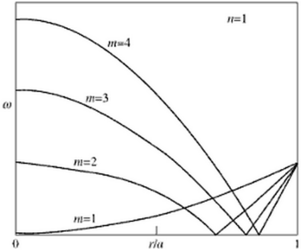
This equation has a continuous spectrum of solutions, and becomes singular where

$$\rho_0\omega^2 - F^2 = 0 \qquad m - nq = \pm\frac{\omega r}{B_0/\sqrt{\mu_0\rho_0}}$$

Alfvén Eigenmodes in a cylinder

For a given m, n, the frequency ω therefore varies with radius:

$$\omega(r) = \pm\frac{V_A}{r}(m - nq)$$

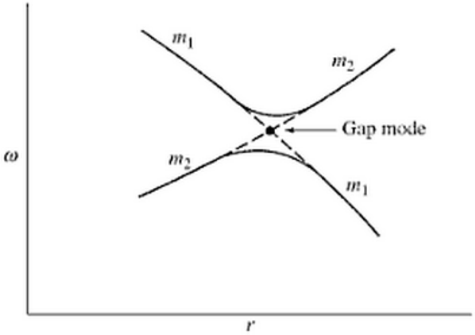


- Waves can excite other nearby waves with similar phase velocities
- These eventually destructively interfere, damping the mode
- This is called **continuum damping**

Alfvén Eigenmodes in a tokamak

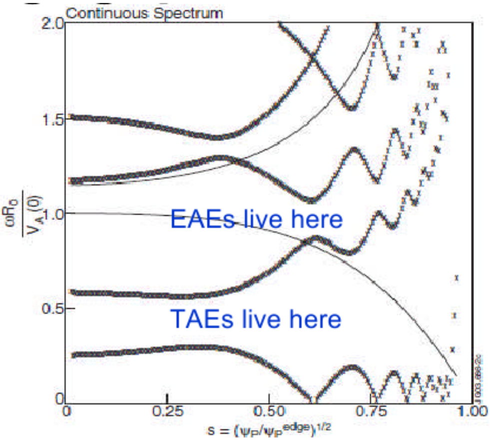
In a tokamak the Alfvén eigenmodes are similar, except that the poloidal harmonics are no longer independent: toroidal shaping couples m to $m+1$ because $B_0 \sim 1 + \frac{r}{R}\cos\theta$.

Where modes with different m have the same frequency, toroidal coupling changes the solutions:

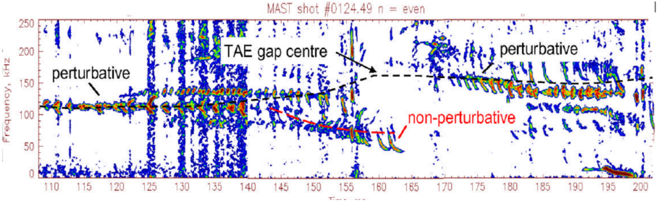


Alfvén Eigenmodes in a tokamak

- In between the two branches there's a single "gap" mode
- Called **Toroidal Alfvén Eigenmodes** (TAE)
- Coupling due to plasma shaping creates additional gap modes e.g. **Elipticity induced Alfvén Eigenmode** (EAE)



- Because this mode is not part of a continuum, it is only weakly damped
- Can be driven by a super-Alfvénic particle population from e.g. NBI or fusion reactions



- Perturbative modes follow TAE gap predicted by MHD
- Nonperturbative mode frequency not given by MHD

MAST results from M.P.Gryaznevich & S.E.Sharapov, Nucl. Fusion **46** S942 (2006)

Summary

- Small amplitude perturbations can be used to study waves
- In a uniform plasma, MHD predicts three wave branches: Shear Alfvén, and the fast and slow magnetosonic waves
- Variation in equilibrium quantities gives a continuum of modes, which experience damping by spreading energy between themselves
- Toroidicity couples poloidal modes together, creating gap modes called Toroidal Alfvén Eigenmodes (TAEs)
- These can be driven unstable by resonant interactions with fast particles, and are only weakly damped
- Various damping and drive mechanisms not described by MHD
- Non-perturbative modes are a very hard problem...