Images of hyperbolic groups

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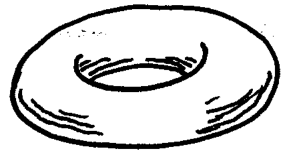
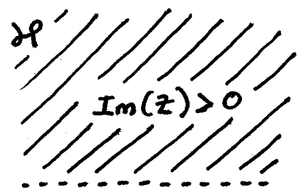
E elliptic curve over $\mathbb{Q}$:  $\subseteq \mathbb{CP}^2$

$$Y^2 Z = 4X^3 + aXZ^2 + bZ^3$$
$$a, b \in \mathbb{Q}$$

Taniyama-Shimura:

$E \cong (\mathscr{H} \cup \mathbb{QP}^1)/\Gamma$,



$\Gamma \leq \mathrm{SL}_2\mathbb{Z}$ defined by congruences:
there is $m \in \mathbb{Z}^{>0}$, $A_1, \ldots, A_k \in \mathrm{SL}_2\mathbb{Z}$
with $A_1 = \mathrm{Id}$, such that

$\Gamma = \{A \in \mathrm{SL}_2\mathbb{Z} : A \equiv A_i \pmod{m} \text{ for some } i\}$.

$\Rightarrow$ FLT

obstruction: is every $\Gamma \leq_n \mathrm{SL}_2\mathbb{Z}$ defined by congruences?

Weil-Belyi $\Rightarrow$

$E \cong (\mathscr{H} \cup \mathbb{QP}^1)/\Gamma$ where $\Gamma \leq_n \mathrm{SL}_2\mathbb{Z}$

Arithmetic groups and congruence subgroups

$V_{\mathbb{Q}}$ a $\mathbb{Q}$-space, $V = V_{\mathbb{Q}} \otimes \mathbb{R}$.

$\mathbb{G} \subseteq GL(V)$ a real algebraic $\mathbb{Q}$-group.

Λ ($\cong \mathbb{Z}^n$) free $\mathbb{Z}$-module in $V_{\mathbb{Q}}$,

$$\mathbb{G}_\Lambda = \{g \in \mathbb{G} : g(\Lambda) = \Lambda\}.$$

$\Gamma \leq \mathbb{G}$ arithmetic $\Leftrightarrow \Gamma$ commensurable with $\mathbb{G}_\Lambda$ for some Λ.

$\nearrow$ $\Gamma \cap \mathbb{G}_\Lambda$ finite index in Γ and $\mathbb{G}_\Lambda$

$\Gamma \leq G$ semisimple real Lie group ($< \infty$ components) is *arithmetic* $\Leftrightarrow$ there is a semisimple $\mathbb{G}$ as above, and

$$\begin{array}{ccc} \tilde{\mathbb{G}}^o & \xrightarrow{\text{epimorphism } \psi} & G^o \\ {\scriptstyle p}\downarrow {\scriptstyle \text{covering}} & & \\ \mathbb{G}^o & & \end{array}$$

(1). $\ker \psi$ compact;
(2). $\psi(p^{-1}(\mathbb{G}^o_\Lambda))$ commensurable with Γ.

Γ arithmetic; $\Gamma(m) = \{\gamma \in \Gamma : \gamma \equiv \text{Id} \pmod{m}\}$;
A subgroup of Γ is *congruence* iff contains a $\Gamma(m)$ for some m.

Congruence Subgroup Problem. Let $\Gamma \subseteq G$ be arithmetic. Are all finite index subgroups of Γ congruence subgroups?

Many positive solutions for $\Gamma \subseteq G$, $\mathrm{rk}_{\mathbb{R}} G > 1$.

[Serre, Bass, Milnor, Matsumoto, Raghunathan, Platinov, ...]

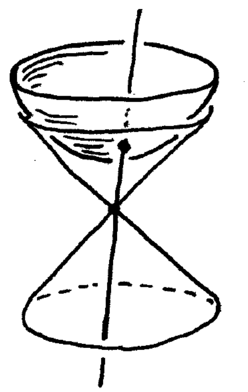
quaternions

On the other hand: $V_{\mathbb{K}}$ $(n+1)$-dim. space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

$B(\boldsymbol{u}, \boldsymbol{v}) = -u_1\overline{v_1} + \sum_{i \geq 2} u_i \overline{v_i}$ Hermitian form signature $(1, n)$.

$\mathbb{K}H^n = \{\boldsymbol{v} \in V_{\mathbb{K}} \,|\, B(\boldsymbol{v}, \boldsymbol{v}) < 0\}/\boldsymbol{v} \sim \lambda\boldsymbol{v}$,
("Projectivised time-like vectors")
gives n-dim. $\mathbb{K}$-hyperbolic space.

eg: $\mathbb{K} = \mathbb{R}, n = 2$:



$\mathbb{R}H^n (= \mathrm{JI}^n), \mathbb{C}H^n, \mathbb{H}H^n$
$+\mathbb{O}H^n$ (octave hyperbolic plane)...

... the ($\mathbb{R}$-) rank 1 symmetric spaces.

$\mathrm{SU}_{1,n}(\mathbb{K}) = f \in \mathrm{SL}(V_{\mathbb{K}})$ preserving form B,

... gives $\mathbb{R}$-rank 1 simple groups,

$$\mathrm{SO}_{1,n}, \mathrm{SU}_{1,n}, \mathrm{Sp}_{1,n} \text{ and } \mathrm{F}_4.$$

Some negative solutions ($\text{rk}_{\mathbb{R}} G = 1$)

Conjecture (Serre 1970). Γ arithmetic $\subseteq G$ simple, $\text{rk}_{\mathbb{R}} G = 1$, then Γ *fails* to have the congruence subgroup property.

Millson's property: $\Gamma' \leq \Gamma$ with $b_1 = \text{rk}_{\mathbb{Z}} H_1(\Gamma') \neq 0 \Rightarrow \Gamma$ fails CSP.

(i). arithmetic $\Gamma \subseteq \text{SO}_{1,n}(\mathbb{R}), n \neq 3, 7$, then all have Millson's property;

[Millson, Li, Raghunathan, Venkataramana, Lubotzky, $\ldots$]

(ii). arithmetic $\Gamma \subseteq \text{SU}_{1,n}(\mathbb{C})$, all have Millson's property;

[Kazhdan, Shimura, Borel, Wallach]

(iii). $\Gamma \subseteq$ other $\text{rk}_{\mathbb{R}} = 1$ groups $\Rightarrow \text{rk}_{\mathbb{Z}} H_1(\Gamma) = 0$.

Subgroup growth: If Γ to have congruence subgroup property then asymptotically,

$$\log \underset{\substack{\nearrow \\ \text{number of subgroups of index } \leq n}}{\sigma_n(\Gamma)} \sim \frac{\log n}{\log \log n}$$

But, eg: if $\Gamma \subseteq \text{SO}_{1,n}, n = 2, 3$ then $\sigma_n(\Gamma)$ grows quicker than this!

Property (A). Γ surjects infinitely many alternating groups A_n.

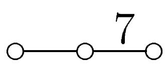
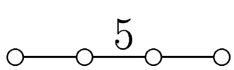
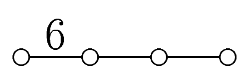
. . . arose historically from . . .

The Hurwitz Problem: M orientable compact (resp. non-compact, finite volume) $\mathbb{R}H^n$-manifold; then

$$|\mathrm{Aut}^+(M)| \le \frac{\mathrm{vol}(M)}{\mathrm{vol}(\mathbb{R}H^n/\Gamma)}, \tag{1}$$

$\Gamma =$ uniform (resp. non-uniform) lattice in $\mathrm{Isom}^+\mathbb{R}H^n$ of smallest volume.

Which finite (simple) G = $\mathrm{Aut}(M)$ for M acheiving (1)? $\Leftrightarrow$ Which finite (simple) G arise as $1 \to \pi_1(M) \to \Gamma \to G \to 1$?

	compact	non-compact
$n = 2$	$\Gamma =$ triangle group  "classical Hurwitz problem" [Conder]	$\Gamma = \mathrm{PSL}_2\mathbb{Z} \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ "$(2,3)$-generation problem" [Miller]
$n = 3$	$\Gamma = (\mathbb{Z}/2\mathbb{Z}) \ltimes$ tetrahedral group  [Everitt]	$\Gamma =$ tetrahedral group  [Everitt]
$n \ge 4$	?	?

Some groups with property (A)

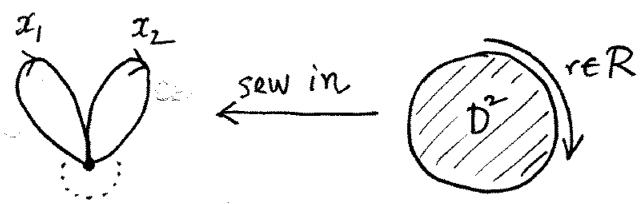
[Pyber-Müller] $A * B$ for A, B finite, non-trivial, not both $\cong \mathbb{Z}/2\mathbb{Z}$, surjects almost all A_n.

[Everitt, conjectured by G. Higman c1969] Every lattice in $\mathrm{PSL}_2\mathbb{R}$ ($\approx \mathrm{SO}_{1,2}(\mathbb{R})$) surjects almost all A_n.

$\Rightarrow$ eg: given $p, q, r \in \mathbb{Z}^{>0}$ prime, with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, there is $N \in \mathbb{Z}^{>0}$ such that A_n is (p, q, r)-generated for all $n \geq N$.

To show $\Gamma = \langle X; R \rangle$ *has property (A)*:

$$\Gamma \to \mathrm{Sym}\,\underset{\nwarrow \text{finite}}{\Omega} \quad \leftrightarrows \quad \begin{matrix} K \\ \downarrow \\ K_0 \end{matrix} \quad \text{finite covering of } CW\text{-complexes}\ldots$$

...where $\pi_1(K_0) \cong \Gamma$, eg: 

Look for complexes that can be "pasted" together:

$$\left\{ \begin{matrix} K_i \\ \downarrow \\ K_0 \end{matrix} \right\}_{i=1\ldots n} \longrightarrow \begin{matrix} [\![K_1 \ldots K_n]\!] \\ \downarrow \\ K_0 \end{matrix}$$

$$\{\Gamma \to \mathrm{Sym}\,\Omega_i\}_{i=1\ldots n} \qquad \Gamma \to \mathrm{Sym}\,(\cup_i \Omega_i)$$

Finally, use classical "recognition theorems" for A_n.

$\Gamma \subseteq G$ arithmetic with property (A) $\Rightarrow$ Γ *fails* to have the congruence subgroup property.

Question. Does every lattice $\Gamma \subseteq G$, simple, $\mathrm{rk}_{\mathbb{R}} = 1$ have property (A)?

		(A)	CSP
"unambigously" rank 1	$\mathrm{SO}_{1,2}(\mathbb{R})$	all Γ	all fail
	$\mathrm{SO}_{1,n}(\mathbb{R}), n \geq 3$	some examples	all fail*
	$\mathrm{SU}_{1,n}(\mathbb{C})$	?	all fail
"rank 2-ish" eg: property $(T)\ldots$	$\mathrm{Sp}_{1,n}(\mathbb{H})$	?	?
	F_4	?	?

(* some exceptions when $n = 7$)