# Weyl groups, lattices and geometric manifolds

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arXiv:math.GR/0706???

## Executive summary

- $X = S^n, \mathbb{E}^n, \mathbb{H}^n$

- Aim: find explicit examples of $X$-manifolds (with small volume).

- [Killing-Hopf]:

$X$-manifolds = Clifford-Klein space forms $X/\Pi$

$\Pi$ acting properly, freely, by isometries

$= X/\Pi$ with $\Pi \subset \text{Isom}(X)$ discrete, torsion free.

$\uparrow$

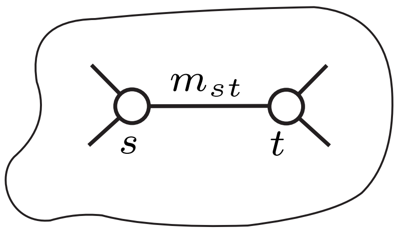
$(X \neq S^n)$

## Coxeter groups

- $(W, S) = \langle s \in S \,|\, (st)^{m_{st}} = 1 \rangle$

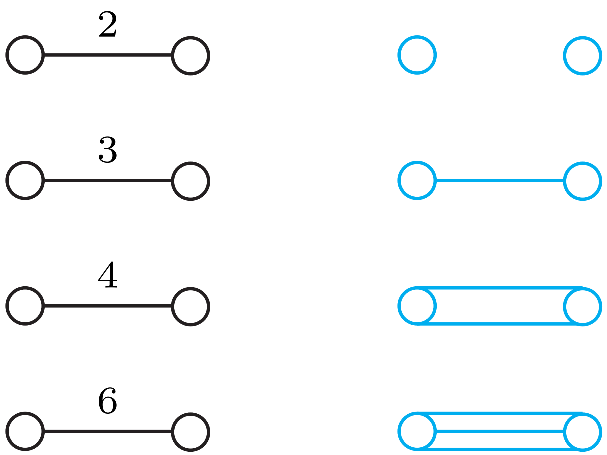
$$m_{st} = m_{ts} \in \mathbb{Z}^{\geq 1} \cup \{\infty\}$$
$$m_{st} = 1 \Leftrightarrow s = t$$

- Coxeter symbol $\Gamma$:

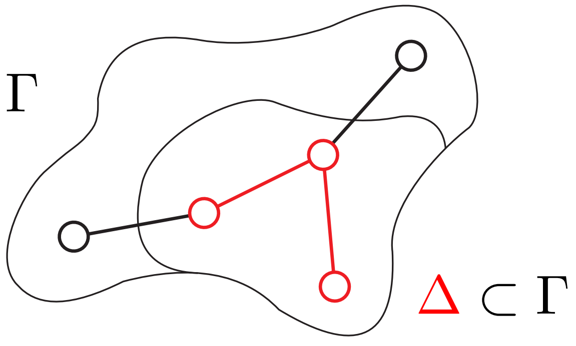



$|\Gamma| :=$ rank $W$
write $W(\Gamma) := (W, S)$

Coxeter Dynkin



## Coxeter groups

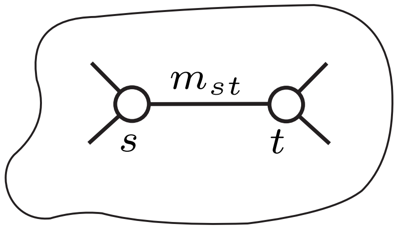



$W(\Delta) := \langle s \in \Delta \rangle$

visible subgroup

## Coxeter groups

Coxeter groups are (real) reflection groups!



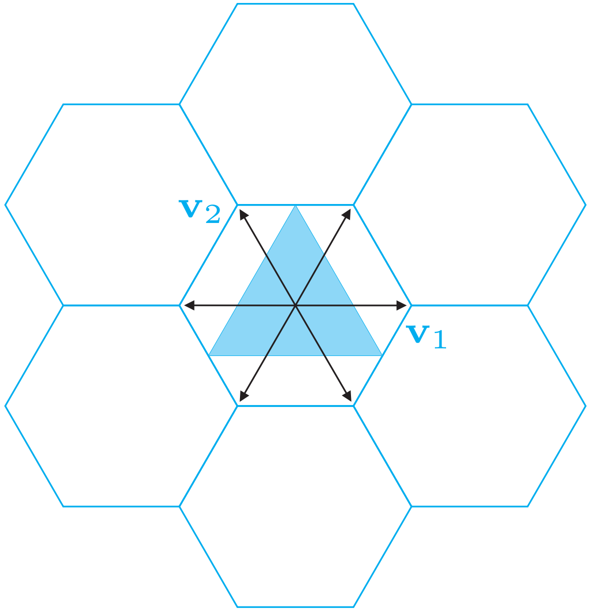

$$V := \langle \mathbf{v}_s \mid s \in S \rangle_{\mathbb{R}}$$

$$B(\mathbf{v}_s, \mathbf{v}_t) := -\cos \frac{\pi}{m_{st}}$$

$$\sigma_s(\mathbf{u}) = \mathbf{u} - 2B(\mathbf{u}, \mathbf{v}_s)\mathbf{v}_s$$ reflection

$s \mapsto \sigma_s$ gives $W(\Gamma) \to \mathrm{GL}(V)$ reflectional representation

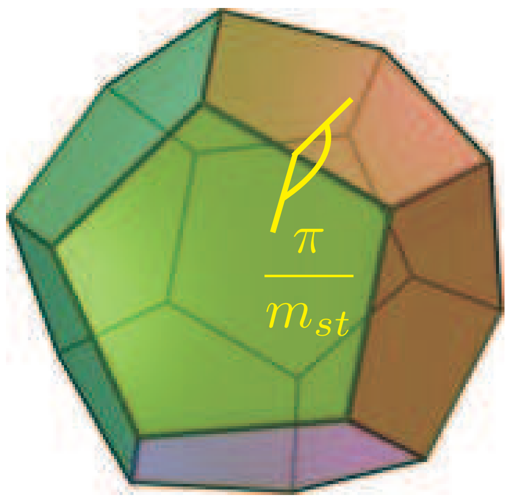
- Eg: $\Gamma =$ 

## Coxeter groups

- $X = S^n, \mathbb{E}^n, \mathbb{H}^n$: Coxeter polytope $P \subset X$

$W = \langle$reflections in $P\rangle \cong$ a Coxeter group

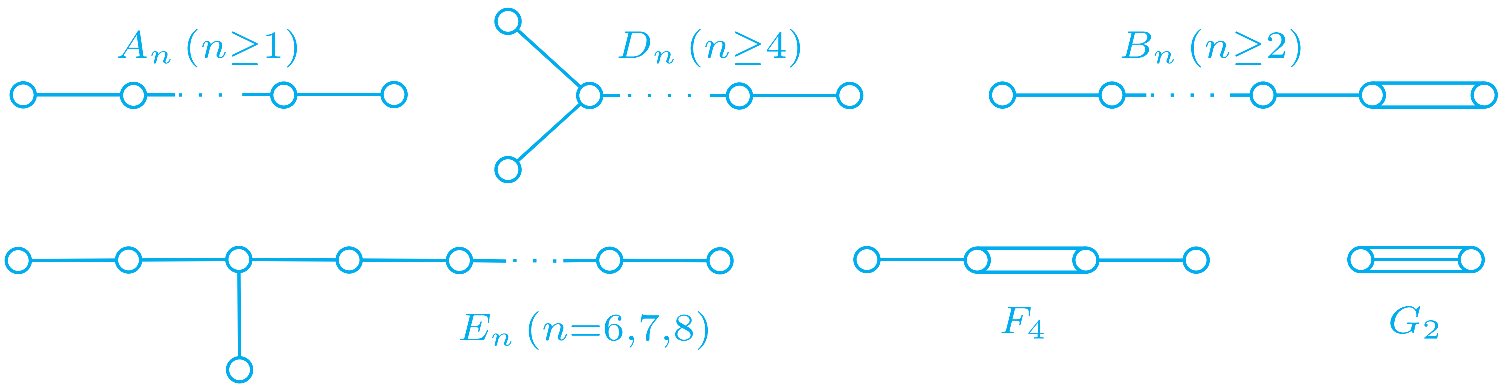


## Weyl groups

- $W(\Gamma)$ Coxeter group, $W(\Gamma) \to \mathrm{GL}(V)$ reflectional representation

- $W(\Gamma)$ a Weyl group $\stackrel{\text{def}}{\Leftrightarrow}$ (i). there is a $W(\Gamma)$-invariant lattice $L \subset V$,
(ii). $W(\Gamma)$ finite.

- $W(\Gamma)$ Weyl group $\Leftrightarrow$ $\Gamma$ disjoint union of:




(all our $\Gamma$ will be connected)

## Weyl groups

- $W(\Gamma)=(W,S)$, $S=\{s_1,\dots,s_n\}$; $w=s_1\dots s_n$ Coxeter element.

- $\Gamma$ tree $\Rightarrow$ all Coxeter elements conjugate;
  Order $h=$ Coxeter number of $W(\Gamma)$.

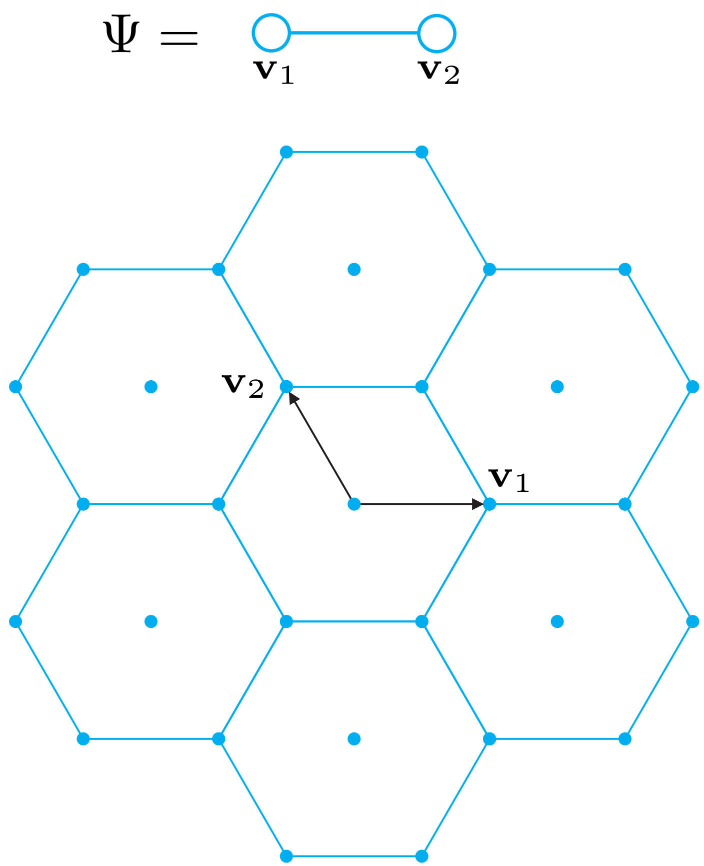
- $W(\Gamma)\to \mathrm{GL}(V)$ reflectional representation; Coxeter elements have eigenvalues

$$\zeta^{m_1},\zeta^{m_2},\dots,\zeta^{m_n}$$

$\zeta=$ primitive $h$-th root of unity; $0\leq m_1\leq\cdots\leq m_n<h$ exponents.

- amazing facts: (i). the $\{m_i+1\}=$ degrees of $W(\Gamma)$.
  (ii). $|W(\Gamma)|=\prod(m_i+1)$.
  $\Big\}$ $W(\Gamma)$ Weyl

## Weyl groups




- root lattice $L := \langle \lambda_i \mathbf{v}_i \rangle_{\mathbb{Z}} \subset V$.

- weight lattice $L \subset \widehat{L}$,

$$\widehat{L} := \left\{ \mathbf{u} \in V \ : \ \frac{2B(\mathbf{u},\mathbf{v})}{B(\mathbf{v},\mathbf{v})} \in \mathbb{Z}, \text{ for } \mathbf{v} \in L \right\}.$$

$$= \langle \mathbf{w}_i \rangle_{\mathbb{Z}} \subset V$$

- $\mathbf{w}_i :=$ simple weight corresponding to $\mathbf{v}_i$, $|\widehat{L}/L| :=$ index of connection

## Weyl groups

$\Psi =$ 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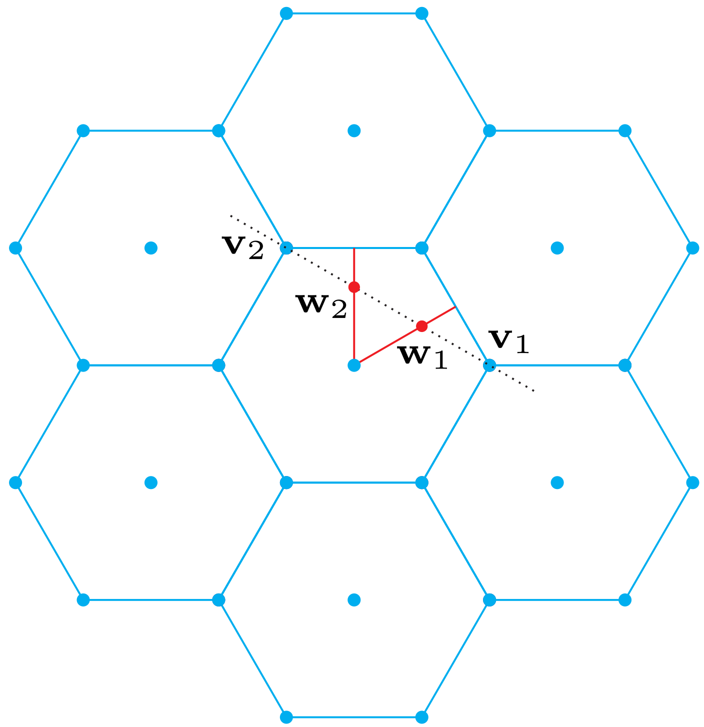


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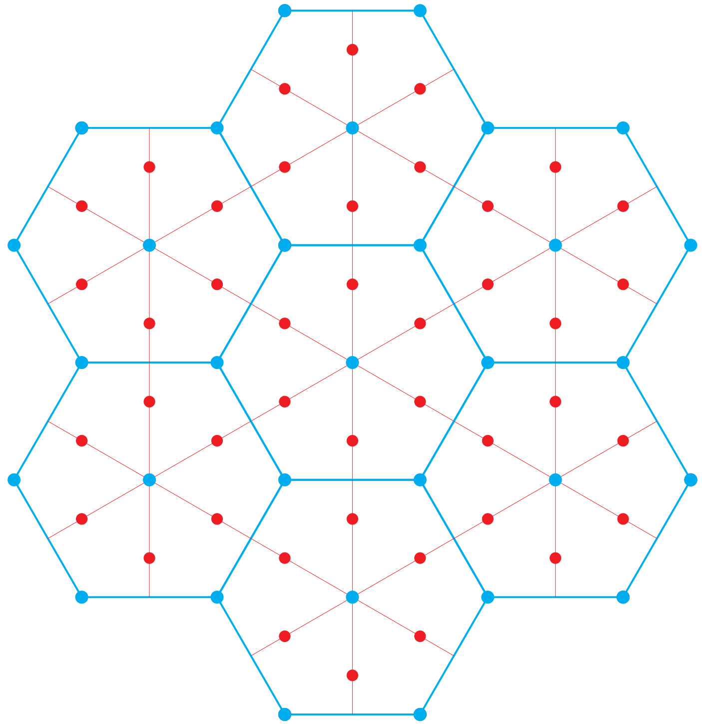

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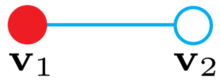
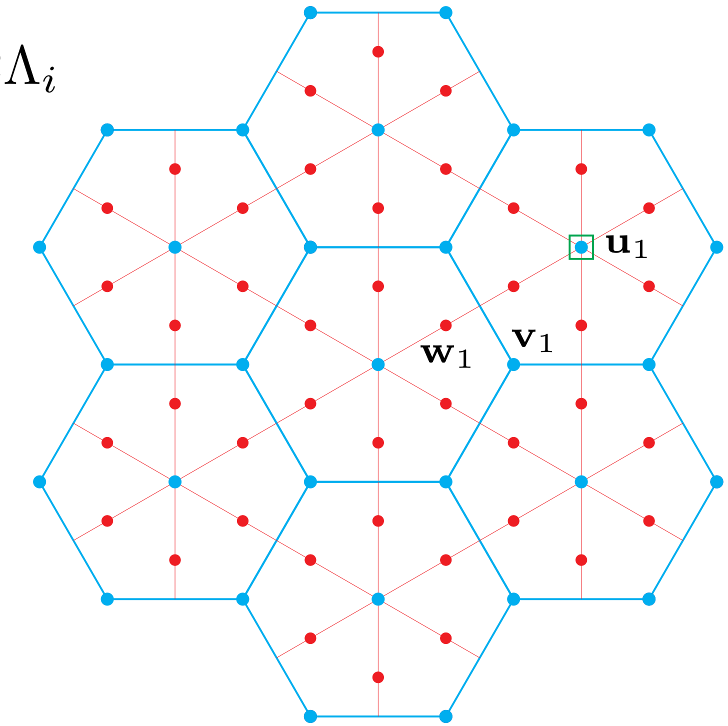
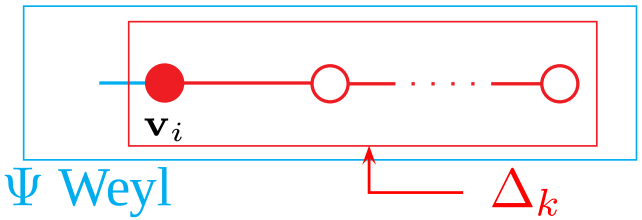
$\Psi =$ 


- $\mathbf{v}_i \in \Psi$, $\mathbf{u}_i := |\widehat{L}/L|\mathbf{w}_i$

$\Lambda_i := \langle \mathbf{u}_i^W \rangle_{\mathbb{Z}} \subset L \qquad \Lambda_i/2 = \Lambda_i/2\Lambda_i$




- 


$W(\Delta_k)$-orbit of $\mathbf{u}_i$ spans subspace $\subset \Lambda_i/2$ of dimension $k+1$ $\Big\}\ (*)$

- $(\Psi, \mathbf{v}_i)$ admissible: $(*)$ all $k$ odd;
  specially admissible: $(*)$ all $k$.

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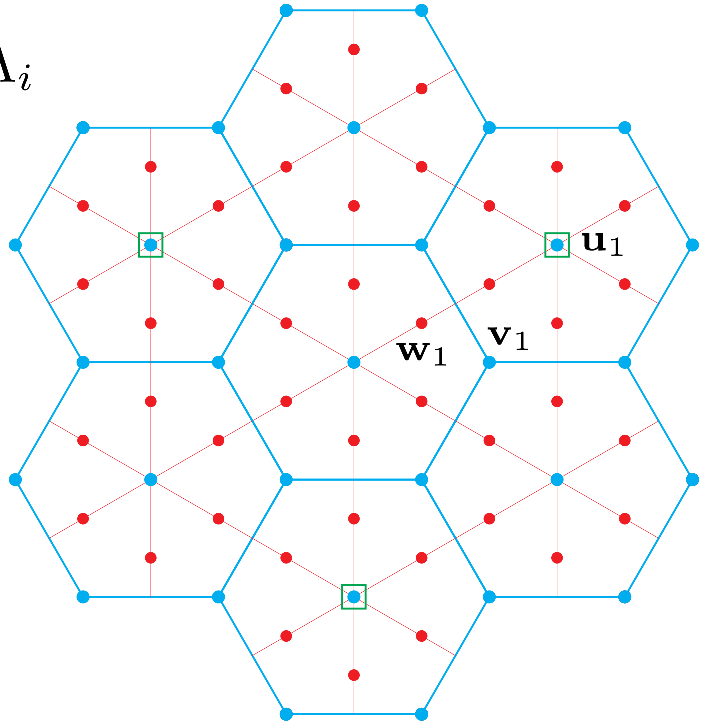
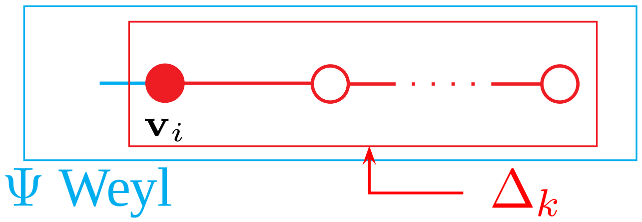



- 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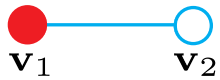
$\Psi =$ 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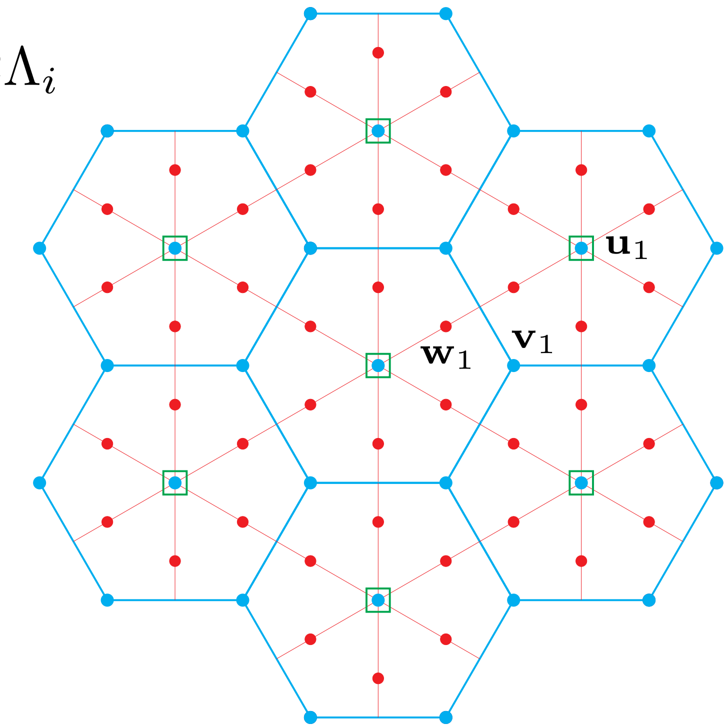
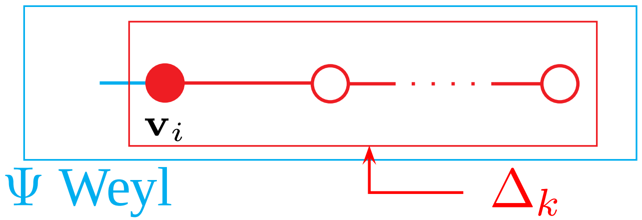



- 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- $(\Psi, \mathbf{v}_i)$ admissible: $(*)$ all $k$ odd;
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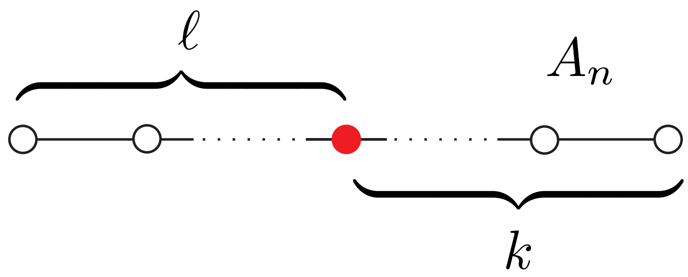
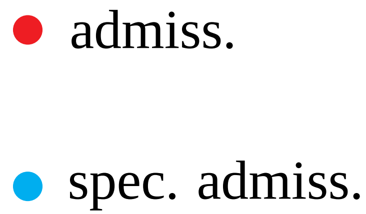
# Weyl groups

- Can classify the admissible and specially admissible pairs $(\Psi, \mathbf{v}_i)$.

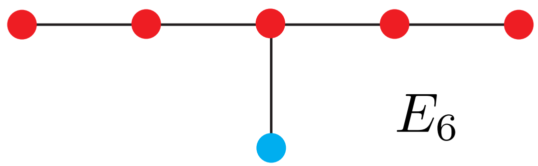
- Eg:

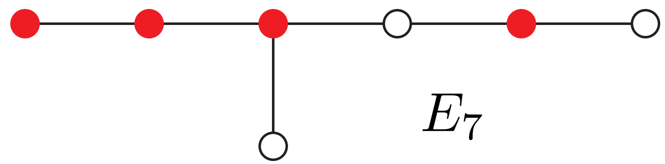


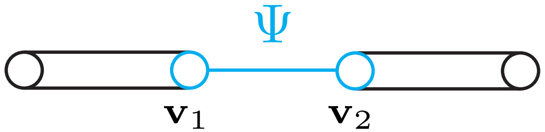

$\ell = 2^a m_1, k = 2^b m_2$ with the $m_i$ odd, and $a < b$

- Eg:

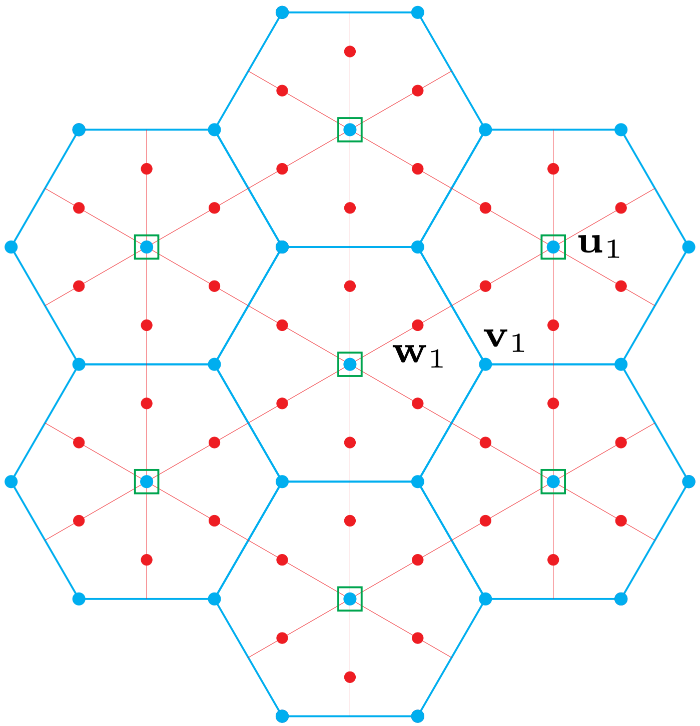


## A homomorphism

- Eg: $\Gamma =$ 


$W = W(\Psi) \to \mathrm{GL}(V)$
reflectional representation




$L =$ root lattice $\subset V$
$\widehat{L} =$ weight lattice $\subset V$

$\mathbf{u}_i = |\widehat{L}/L|\mathbf{w}_i$, $(i = 1, 2)$
$\Lambda_i = \langle \mathbf{u}_i^W \rangle_{\mathbb{Z}} \subset L$

$\Lambda_i/2 = \Lambda_i/2\Lambda_i$

form semi-direct product
$(\Lambda_1/2 \times \Lambda_2/2) \rtimes W(\Psi)$

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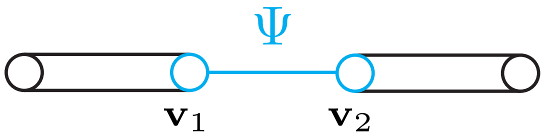
- Eg: $\Gamma =$ 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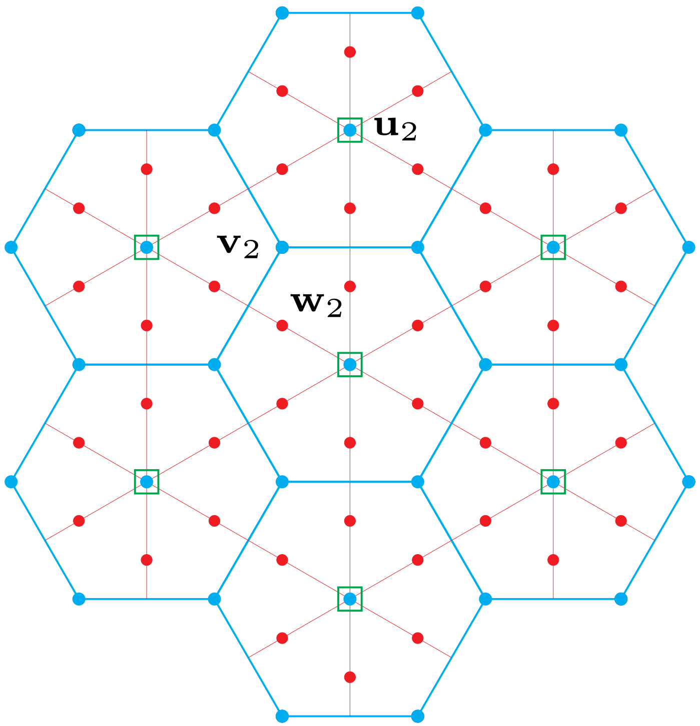


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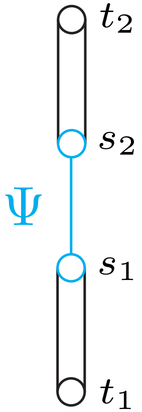
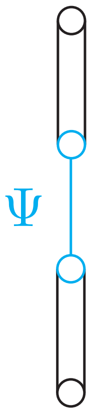
## A homomorphism




$$\left.\begin{array}{l} t_2 \longrightarrow (0,\mathbf{u}_2,1) \\ s_2 \longrightarrow (0,0,s_2) \\ \qquad f_1 \\ s_1 \longrightarrow (0,0,s_1) \\ t_1 \longrightarrow (\mathbf{u}_1,0,1) \end{array}\right\} \in (\Lambda_1/2 \times \Lambda_2/2) \rtimes W(\Psi)$$

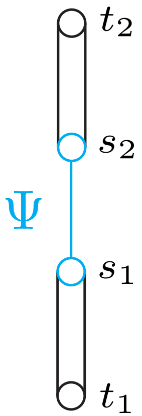



$$g \xrightarrow{f_2} (\#t_1 \text{ in } g \bmod 2, \#t_2 \text{ in } g \bmod 2) \in \mathbb{Z}/2 \times \mathbb{Z}/2$$

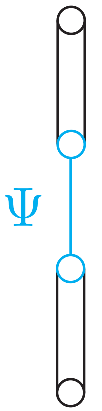
$$\Rightarrow \varphi = f_2 \times f_1 : W(\Gamma) \to (\mathbb{Z}/2)^2 \times (\textstyle\prod \Lambda_i/2 \rtimes W(\Psi))$$

surjective homomorphism

# A homomorphism




$$f_1: \quad t_2 \longrightarrow (0, \mathbf{u}_2, 1), \quad s_2 \longrightarrow (0, 0, s_2), \quad s_1 \longrightarrow (0, 0, s_1), \quad t_1 \longrightarrow (\mathbf{u}_1, 0, 1) \quad \in (\Lambda_1/2 \times \Lambda_2/2) \rtimes W(\Psi)$$

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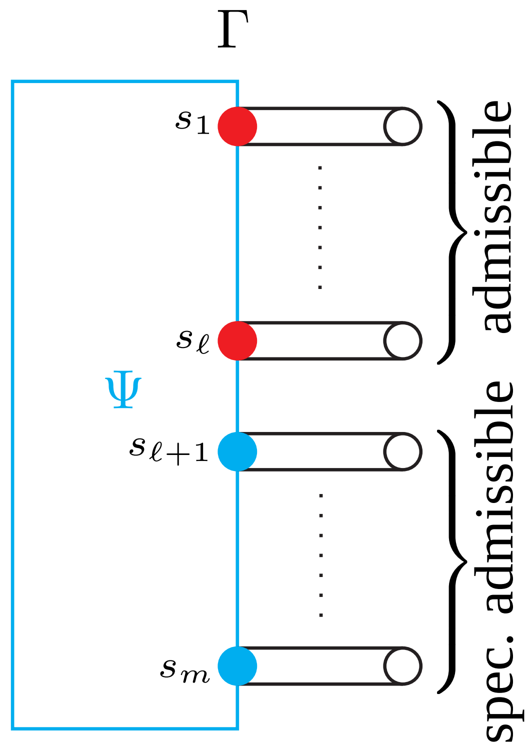
surjective homomorphism with ker $\varphi$ torsion free

Theorem A: $W(\Gamma)$ Coxeter group:

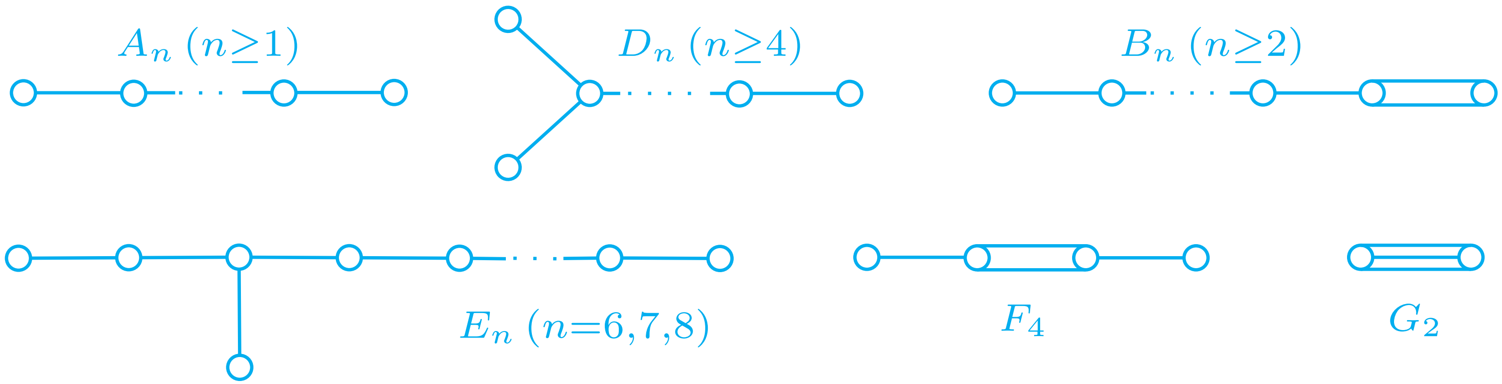
$W(\Psi)$ Weyl group
rank $n$
exponents $m_1, \ldots, m_n$




$\Rightarrow$ have homomorphism

$$\varphi : W(\Gamma) \to (\mathbb{Z}/2)^\ell \times \left( \prod \Lambda_i/2 \rtimes W(\Psi) \right)$$

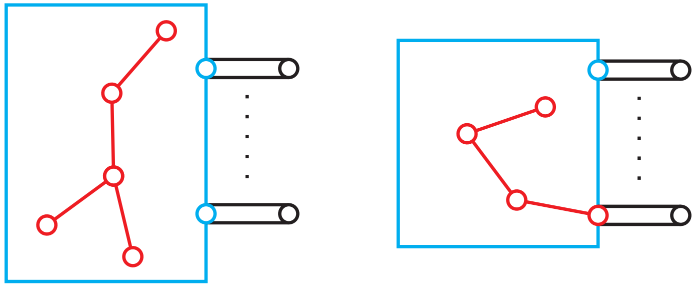
with ker $\varphi$ torsion free index $2^{mn+\ell} \prod (m_i + 1)$

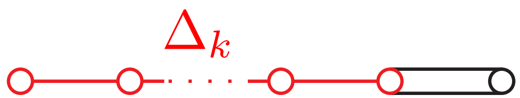


## Theorem A (sketch)

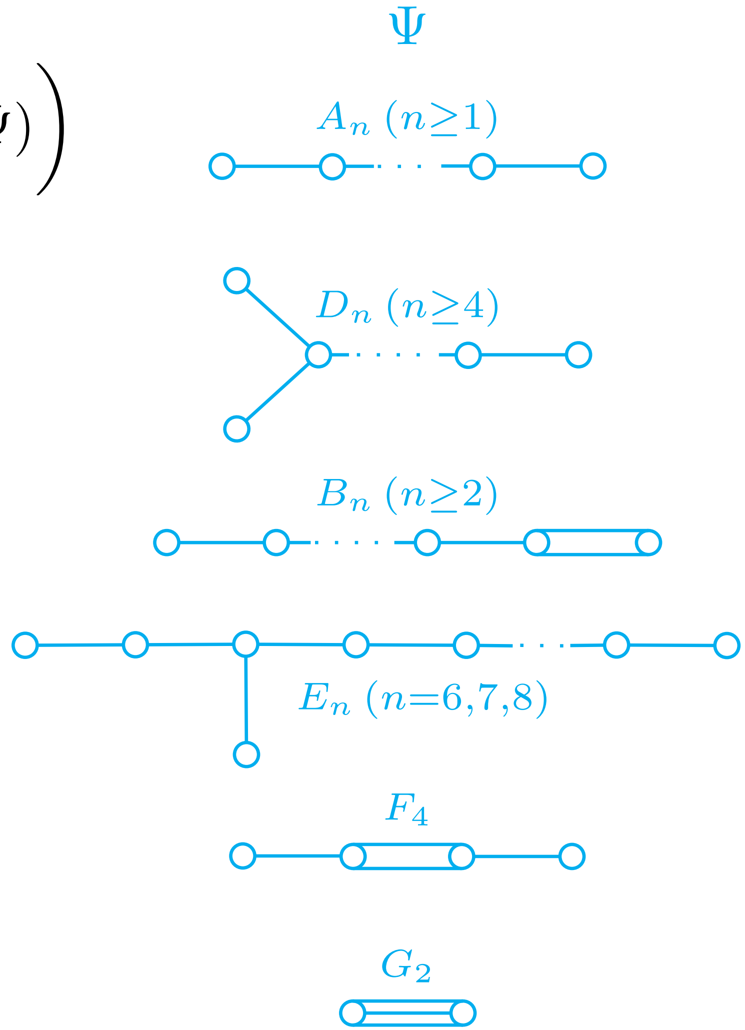
- $W(\Gamma) \xrightarrow{\varphi} (\mathbb{Z}/2)^{\ell} \times \left(\prod \Lambda_i/2 \rtimes W(\Psi)\right)$

with $\ker \varphi$ torsion free:



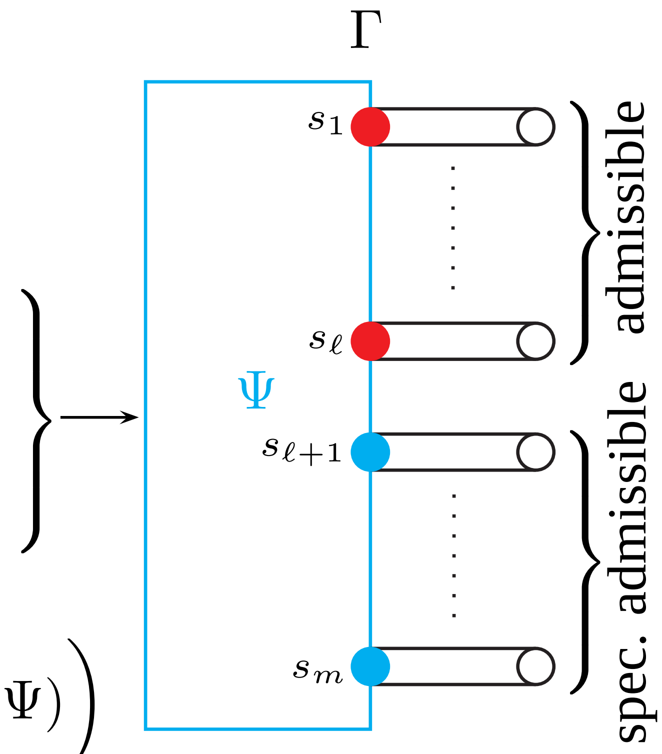



$\cong (\mathbb{Z}/2)^{k+1} \rtimes W(\Delta_k)$



Theorem B: $W(\Gamma)$ Coxeter group

$W(\Psi)$ Weyl group
rank $n$
exponents $m_1, \dots, m_n$
Coxeter number $h = 2^p q$ ($p > 0$, $q$ odd)




$$\varphi : W(\Gamma) \to (\mathbb{Z}/2)^\ell \times \left(\prod \Lambda_i/2 \rtimes W(\Psi)\right)$$

$\Rightarrow$ there is a Coxeter element $\xi \in W(\Psi)$ with

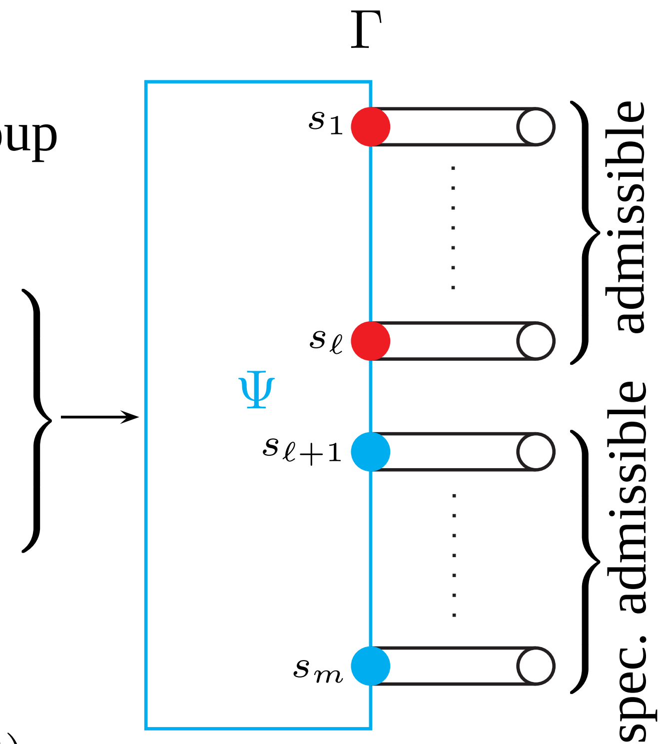
$$(\mathbf{x}, \mathbf{v}, \xi^q) \cong \mathbb{Z}/(2^p) \subset (\mathbb{Z}/2)^\ell \times \left(\prod \Lambda_i/2 \rtimes W(\Psi)\right)$$

and $\varphi^{-1}\mathbb{Z}/(2^p) \subset W(\Gamma)$ torsion free.

## Geometric version

**Theorem**: $W(\Gamma)$ hyperbolic Coxeter group

$W(\Psi)$ Weyl group
rank $n$
exponents $m_1, \dots, m_n$
Coxeter number $h = 2^p q$ ($p > 0$, $q$ odd)




$\Rightarrow$ Galois covering $\widehat{M} \to M$ hyperbolic $N$-manifolds with $\text{Gal}(\widehat{M}, M) \cong \mathbb{Z}/(2^p)$ and

$$\text{vol}(M) = 2^{mn+\ell-p} \prod (m_i + 1) \text{covol} W(\Gamma),$$

where $N \geq n$ is the largest rank of a finite visible subgroup.

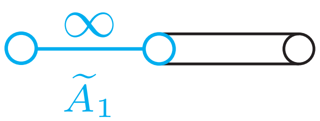
## Example

- $I_{n,1}$ := the odd self-dual Lorentzian lattice rank $n+1$
- $\text{Aut}(I_{n,1})/\text{center} \cong \text{PO}_{n,1}(\mathbb{Z})$ acts cofinitely on $\mathbb{H}^n$.
- [Vinberg-Kaplinskaya] the index $[\text{PO}_{n,1}(\mathbb{Z}) : \langle\text{reflections}\rangle] < \infty$

$$\Leftrightarrow n \leq 19$$

- $\langle\text{reflections}\rangle \cong W(\Gamma)$ with $\Gamma =$

$n = 2$




$n = 3$


$4 \leq n \leq 9$

| $n$ | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|
| $\Psi$ | $A_4$ | $D_5$ | $E_6$ | $E_7$ | $E_8$ | $\widetilde{E}_8$ |

## Example

Galois covering $\widehat{M} \to M$ of hyperbolic $n$-manifolds with $\text{Gal}(\widehat{M}, M) \cong \mathbb{Z}/(2^p)$ and

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| $\Psi$ | $n$ | $m = \ell$ | $h = 2^p q$ | exponents $m_i$ |
|---|---|---|---|---|
| $A_4$ | 4 | 1 | 5 | $1, 2, 3, 4$ |
| $E_6$ | 6 | 1 | $2^2\,3$ | $1, 4, 5, 7, 8, 11$ |
| $E_8$ | 8 | 1 | $2\,3\,5$ | $1, 7, 11, 13, 17, 19, 23, 29$ |

$$\text{vol}(M) = 2^{n-p+1} \frac{(2^{\frac{n}{2}} \pm 1)\pi^{\frac{n}{2}}}{n!} \prod_{i=1}^{n} (m_i + 1) \prod_{k=1}^{\frac{n}{2}} |B_{2k}|.$$

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| $E_6$ | 6 | 1 | $2^2\, 3$ | $1, 4, 5, 7, 8, 11$ |
| $E_8$ | 8 | 1 | $2\, 3\, 5$ | $1, 7, 11, 13, 17, 19, 23, 29$ |

$$\mathrm{vol}(M) = 2^{n-p+1} \frac{(2^{\frac{n}{2}} \pm 1)\pi^{\frac{n}{2}}}{n!} \prod_{i=1}^{n} (m_i + 1) \prod_{k=1}^{\frac{n}{2}} |B_{2k}|.$$

$n = 4$: $\chi(M) = 2$; $n = 6$: $\chi(M) = -2$