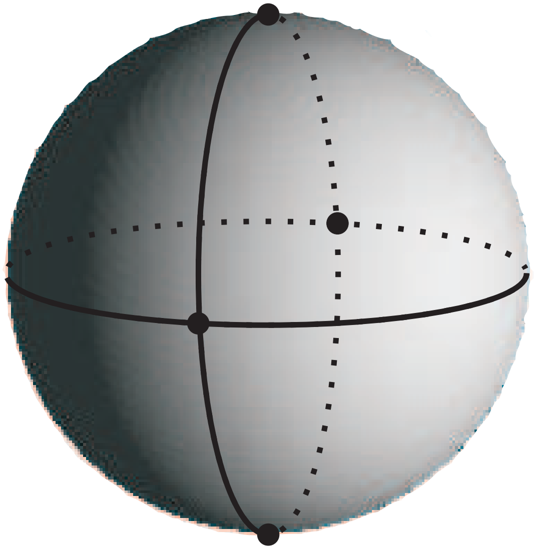
Knots, posets and sheaves

Brent Everitt (York) –joint with Paul Turner (Geneva-Fribourg)

space

$X =$



Euler characteristic:

$$\chi(X) = \sum(-1)^i |X_i|$$

$(= 2)$

homology:

$$H_*(X;\mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$$

0 2

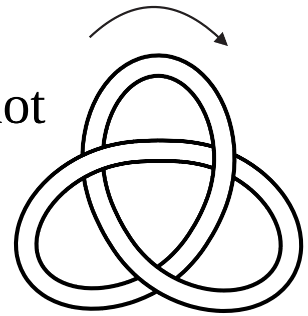
$$\chi = \sum(-1)^i \dim H_i(X)$$

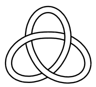
$(= 2)$

$$X \xrightarrow{f} Y \rightsquigarrow H_*(X,\mathbb{Q}) \xrightarrow{f_*} H_*(Y,\mathbb{Q})$$

continuous map homomorphism

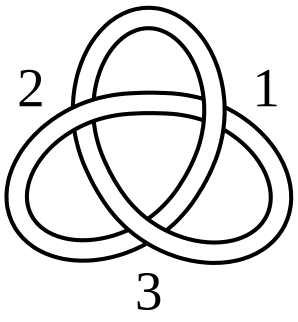
oriented knot

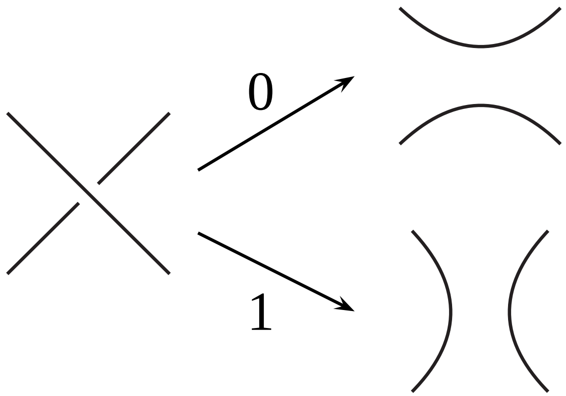
$K =$ 

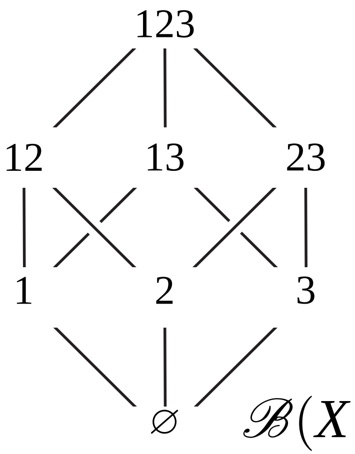
J () $= \dfrac{q^3}{(q+q^{-1})}(-q^6+q^2+1+q^{-2})$

Jones polynomial

?

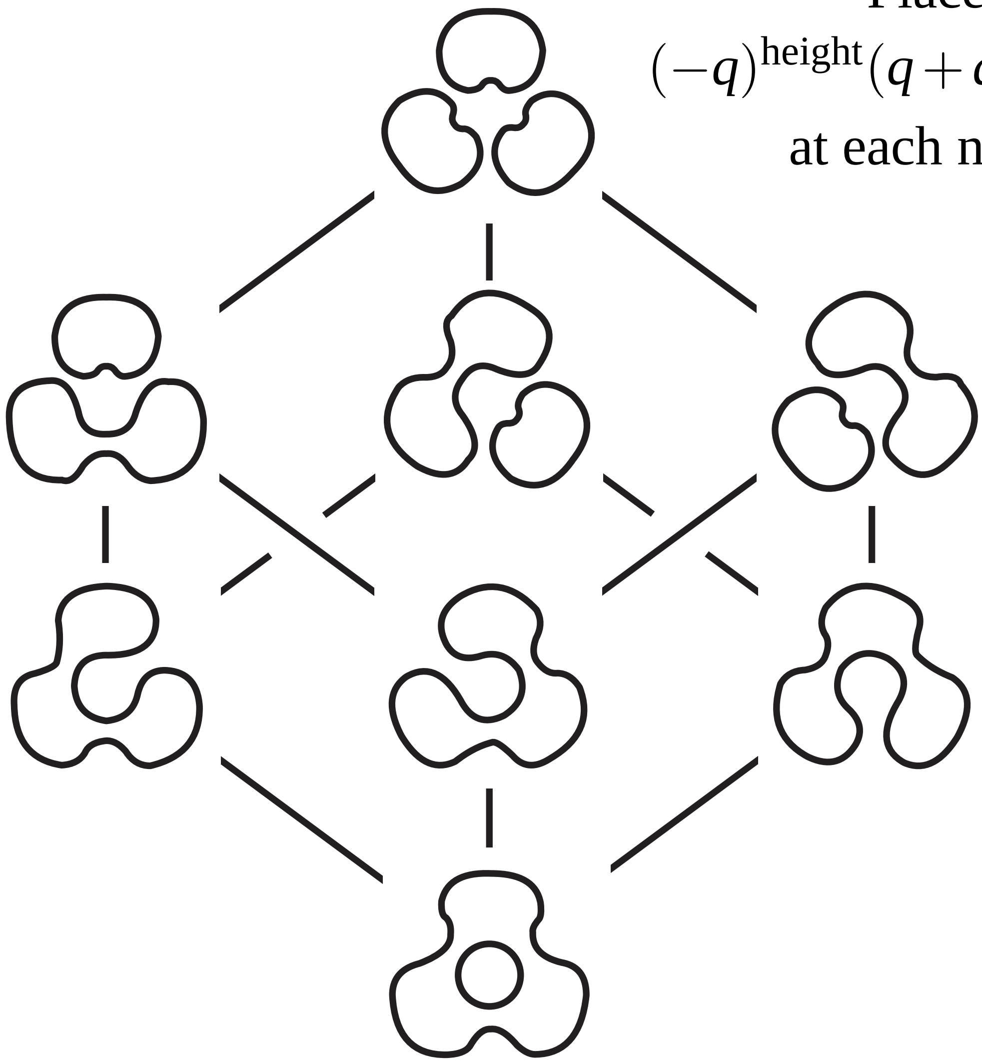


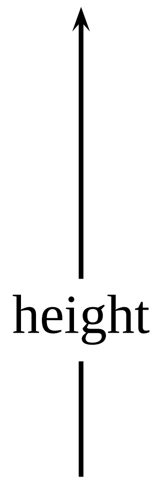


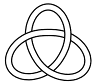
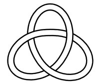
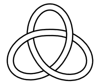


$\mathscr{B}(X), X = \{1,2,3\}$

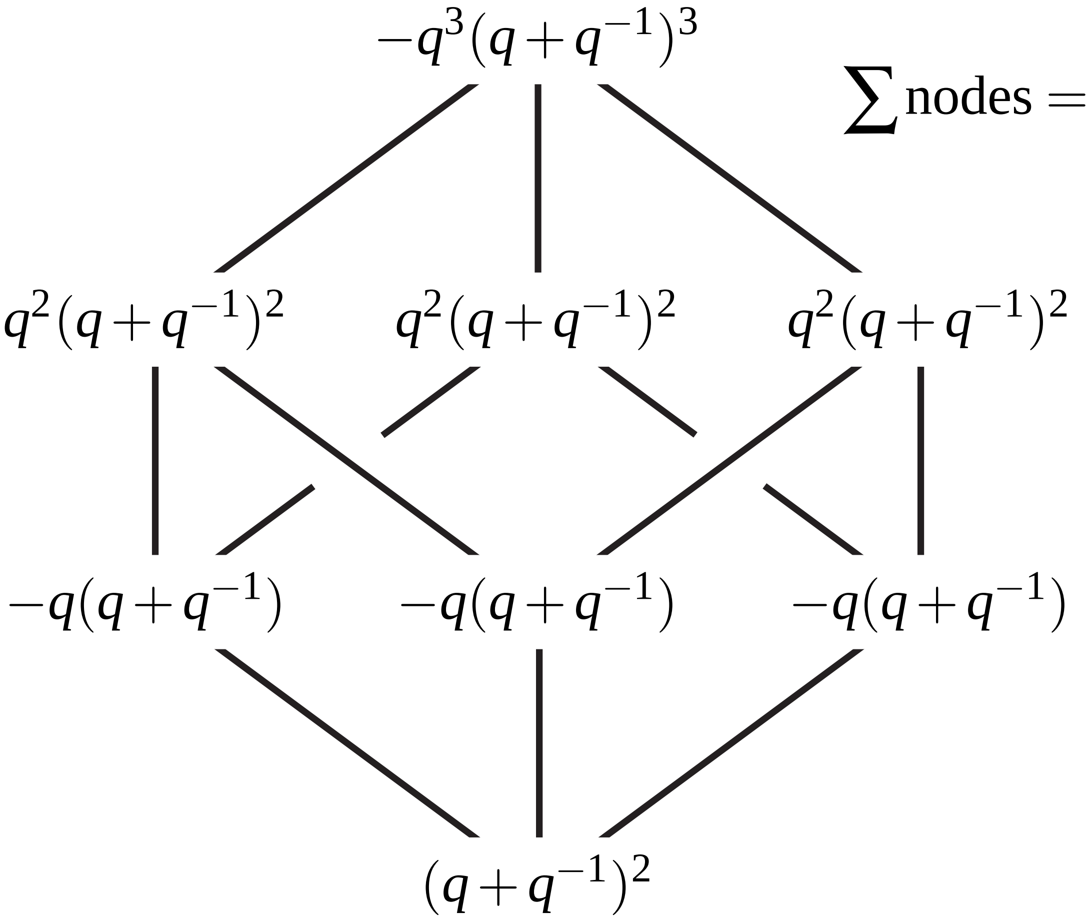
Place
$(-q)^{\text{height}}(q+q^{-1})^{\#\text{circles}}$
at each node

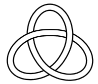


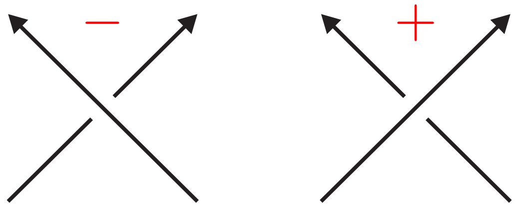


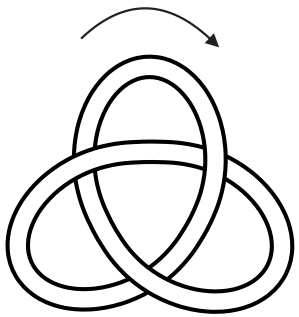
$J\Bigg($  $\Bigg) = \dfrac{1}{(q+q^{-1})}\widehat{J}\Bigg($  $\Bigg) \longleftarrow (-1)^{n_-}q^{n_+-2n_-}\Bigg\langle$  $\Bigg\rangle$

(Jones) (unnormalized Jones) (Kauffman bracket)



$\sum \text{nodes} = \Bigg\langle$  $\Bigg\rangle = -q^6+q^2+1+q^{-2}$





- $A = \bigoplus A_i =$ $\cdots$ | A_{-1} | A_0 | A_1 | $\cdots$ $\quad$ (A_i = vector spaces over k)

$\cdots$	A_{-1}	A_0	A_1	$\cdots$
	-1	0	1	

- direct sum $V \oplus U = \bigoplus (V_i \oplus U_i)$

- tensor product $V \otimes U = \bigoplus (V \otimes U)_k$ with $(V \otimes U)_k = \bigoplus_{i+j=k} V_i \otimes U_j$

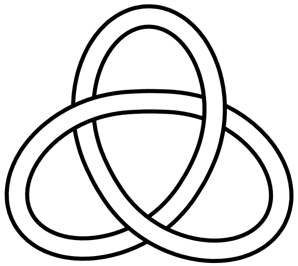
- $V[\ell] =$ (degree shift)

$\cdots$	$A_{-\ell-1}$	$A_{-\ell}$	$A_{\ell+1}$	$\cdots$
	-1	0	1	

- graded dimension $q\dim V := \sum \dim V_j q^j \in \mathbb{Z}[q, q^{-1}]$

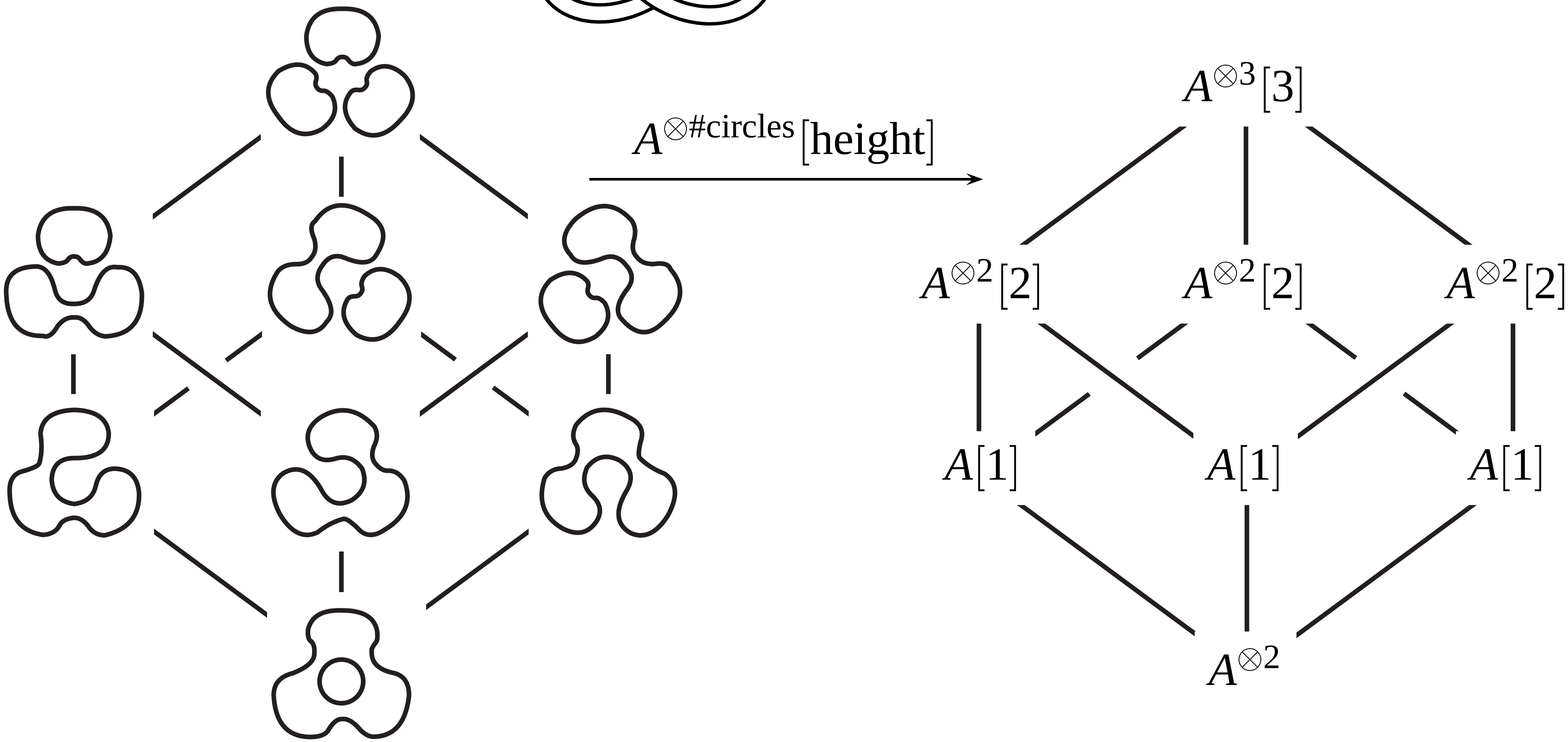
- $q\dim(U \otimes V) = q\dim U \times q\dim V$
  $q\dim V[\ell] = q^\ell \times q\dim V$

Khovanov complex

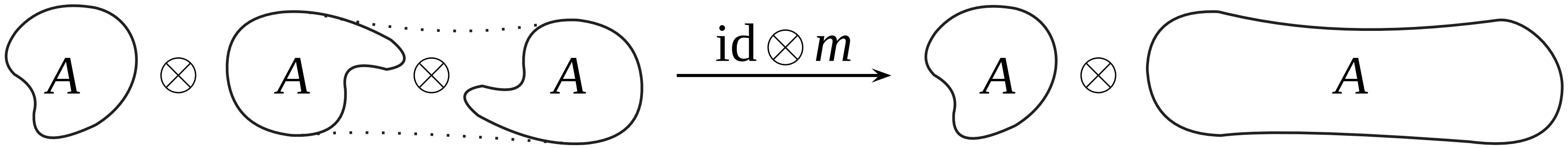


$$A = \mathbb{Q} \oplus \mathbb{Q}$$

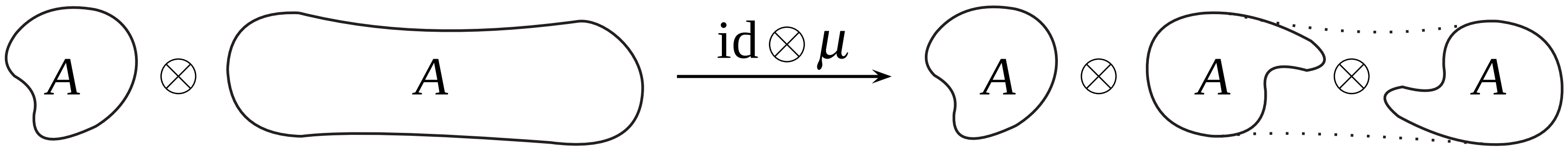
$-1 \quad 1$

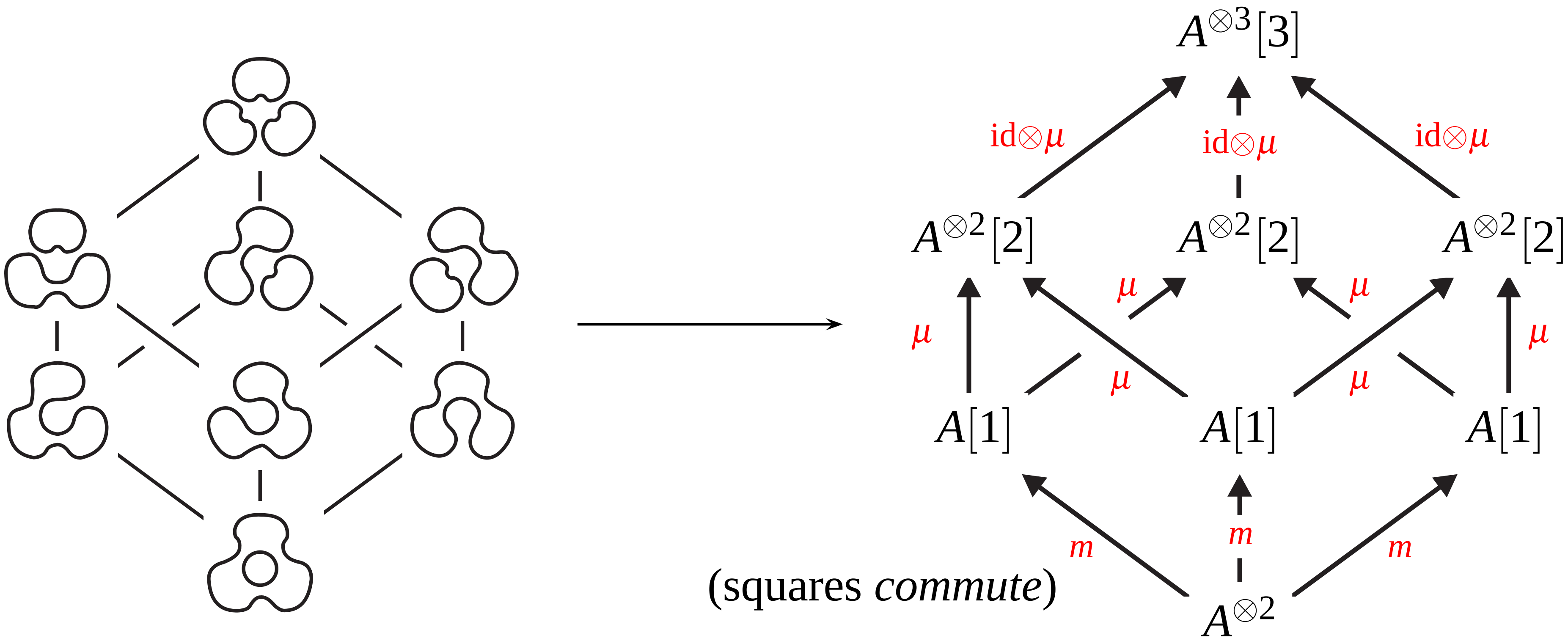


$m : A \otimes A \to A$

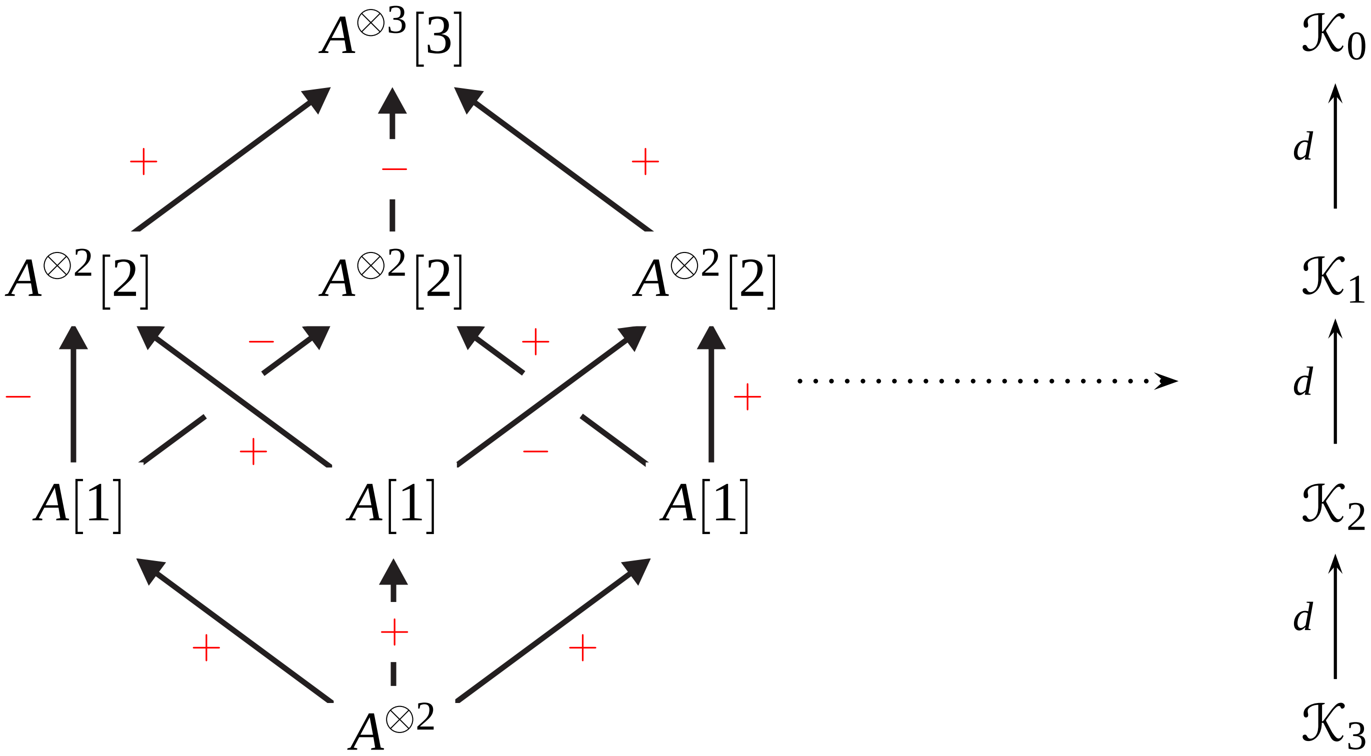


$\mu : A \to A \otimes A$





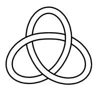
(squares *commute*)

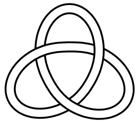


$\mathcal{K}_*$

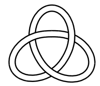
Khovanov (chain) complex

add $\pm$'s to edge maps so squares *anticommute*

Khovanov homology $KH_*\Big($  $, \mathbb{Q}\Big) = H_*(\mathcal{K}_*)$

	6	4	2	0	-2	$q\dim$
KH_0	$\mathbb{Q}$					q^6
KH_1			$\mathbb{Q}$			q^2
KH_2						0
KH_3				$\mathbb{Q}$	$\mathbb{Q}$	$1+q^{-2}$

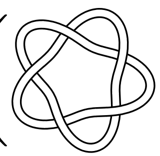
Euler characteristic $\chi(\mathcal{K}_*)$

$= \sum(-1)^i q\dim KH_i \Big($  $, \mathbb{Q}\Big)$

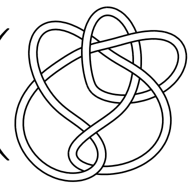
$$= q^6 - q^2 - 1 - q^{-2}$$

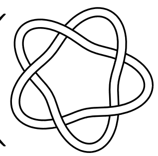
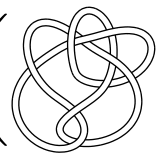
minor miracle: KH_* an invariant (after a bit of nudging)

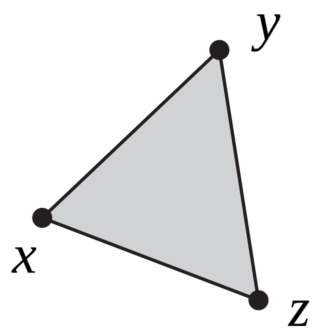
$\mathbb{Q}$						
		$\mathbb{Q}$				
		$\mathbb{Q}$				
				$\mathbb{Q}$		
					$\mathbb{Q}$	$\mathbb{Q}$

$KH_*\Big($  $\Big)$

$\mathbb{Q}$							
		$\mathbb{Q}$					
		$\mathbb{Q}$					
			$\mathbb{Q}$	$\mathbb{Q}$			
			$\mathbb{Q}$		$\mathbb{Q}$		
					$\mathbb{Q}\oplus\mathbb{Q}$		
							$\mathbb{Q}$
						$\mathbb{Q}$	$\mathbb{Q}$

$KH_*\Big($  $\Big)$

- Jones$\Big($  $\Big) =$ Jones$\Big($  $\Big)$
- FUNCTORIAL!!



iff $x < y < z$

- poset $P \longrightarrow |P|$ order (simplicial) complex.

- poset homology = simplicial homology of $|P|$

ie: $H_*(P,R) := H_*(|P|,R)$ = homology of chain complex

$$C_n(P,R) = \bigoplus_{x_0<\cdots<x_n} R$$

with differential $d : C_n(P,R) \to C_{n-1}(P,R)$

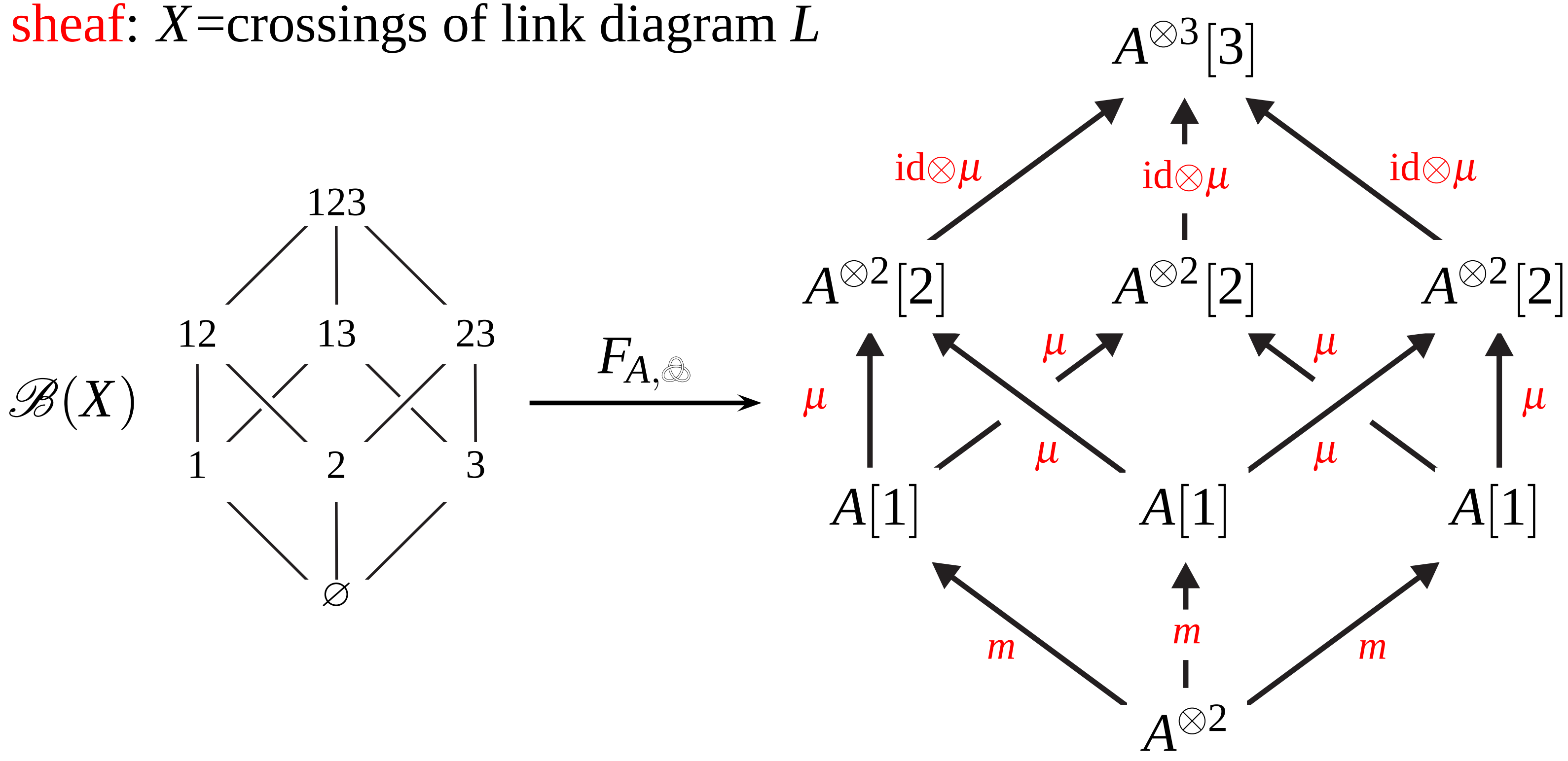
$$\lambda \cdot (x_0 < \cdots < x_n) \overset{d}{\mapsto} \sum_{j=0}^{n} (-1)^j \lambda \cdot (x_0 < \cdots < \widehat{x}_j < \cdots < x_n)$$

- Eg: [Folkman] P finite geometric lattice

$$\widetilde{H}_n(P \setminus \{0,1\},\mathbb{Z}) = \begin{cases} \mathbb{Z}^{|\mu(0,1)|} & n = \text{rk}P - 2, \\ 0 & \text{otherwise.} \end{cases}$$

- $P \xrightarrow{F} R$-**mod** (covariant) functor (= pre-cosheaf of modules over P)

- Eg: Khovanov sheaf: X=crossings of link diagram L



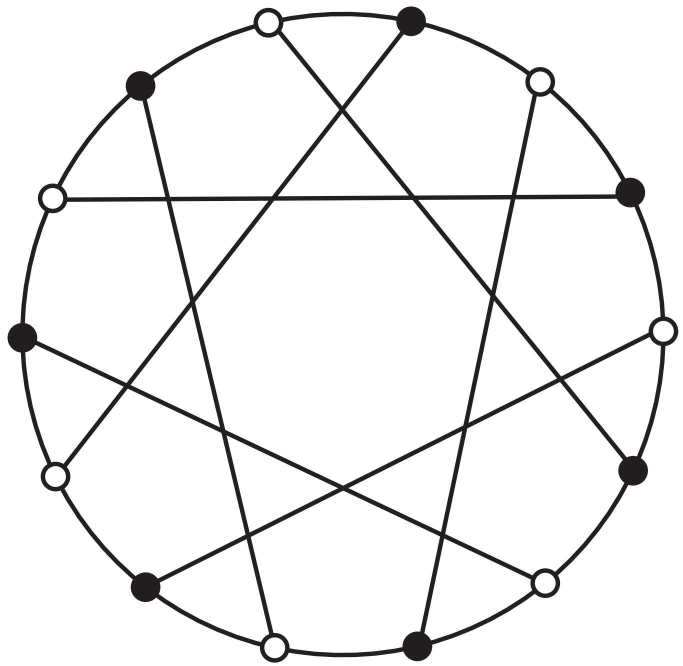
- Eg: sheaf on a building: poset of flags in V (a finite dimensional k-space)

$$(V_1 \subset \cdots \subset V_n) \leq (U_1 \subset \cdots \subset U_m) \Leftrightarrow \text{ each } V_i = \text{ some } U_j$$

sheaf:

$$F(V_1 \subset \cdots \subset V_n) = V_n \text{ and } F[(V_1 \subset \cdots \subset V_n) \leq (U_1 \subset \cdots \subset U_m)] = V_n \hookrightarrow U_m$$

V 3-dimensional over $k = \mathbb{F}_2$



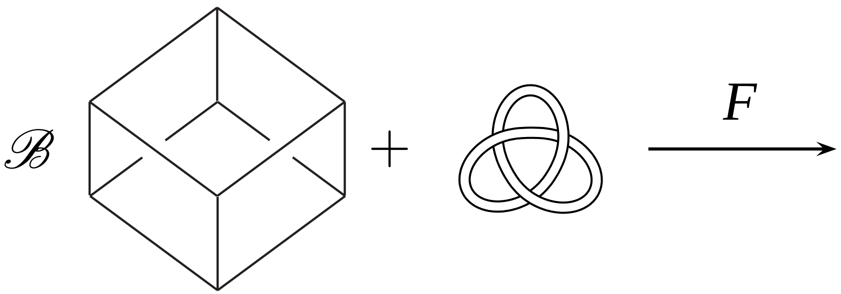
- $P \xrightarrow{F} R$-**mod** sheaf

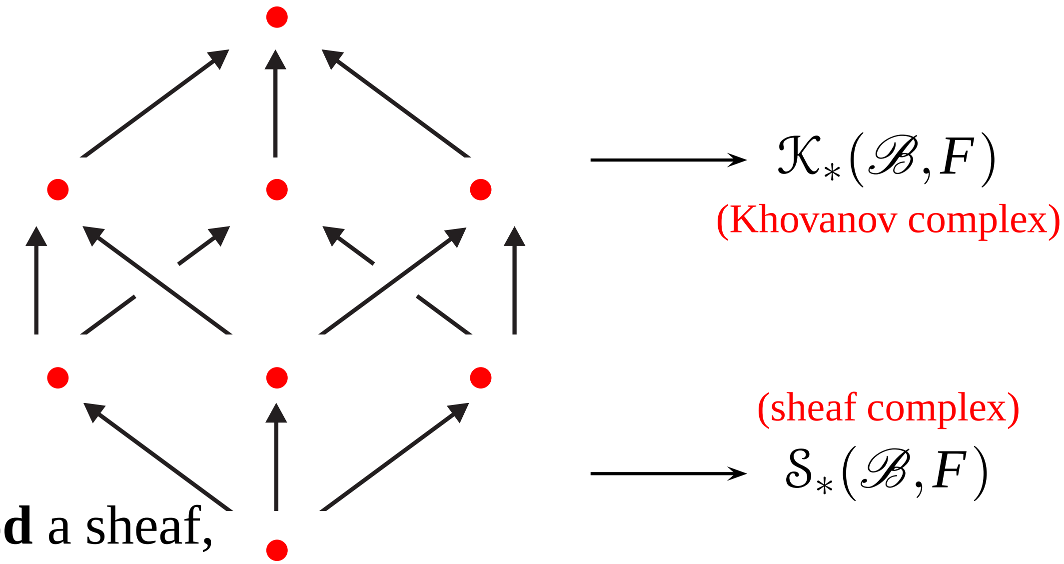
- sheaf homology $\mathcal{H}_*(P,F) =$ homology of chain complex

$$\mathcal{S}_n(P,F) = \bigoplus_{x_0<\cdots<x_n} F(x_0)$$

with differential $d : \mathcal{S}_n(P,F) \to \mathcal{S}_{n-1}(P,F)$

$$\lambda \cdot (x_0 < \cdots < x_n) \overset{d}{\mapsto} F(x_0 < x_1)(\lambda) \cdot (\widehat{x_0} < x_1 < \cdots < x_n)$$
$$+ \sum_{j=1}^{n} (-1)^j \lambda \cdot (x_0 < \cdots < \widehat{x_j} < \cdots < x_n)$$





- **Theorem**: $\mathscr{B}$ Boolean and $\mathscr{B} \stackrel{F}{\to} R$-**mod** a sheaf,

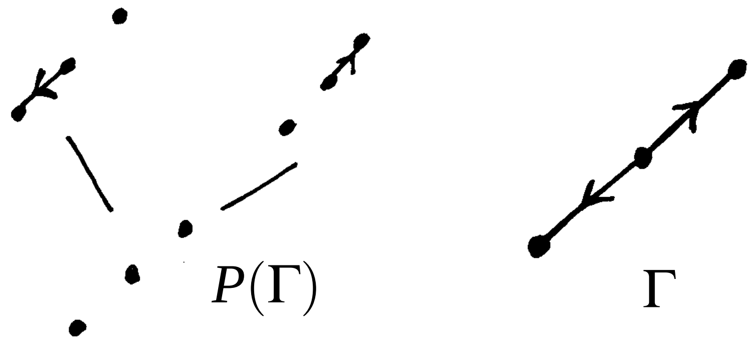
$$KH_*(\mathscr{B},F) \cong \widetilde{\mathcal{H}}_{*-1}(\mathscr{B}\setminus 1,F)$$

- [generally: one can define a "cellular" homology $H_*^{\text{cell}}(P,F)$:

Theorem: P "cellular" poset and $P \stackrel{F}{\to} R$-**mod** a sheaf, then

$$H_*^{\text{cell}}(P,F) \cong \mathcal{H}_*(P,F)$$

Eg: $P =$ geometric lattices, cell posets regular CW-complexes, Cohen-Macaulay posets, ...]

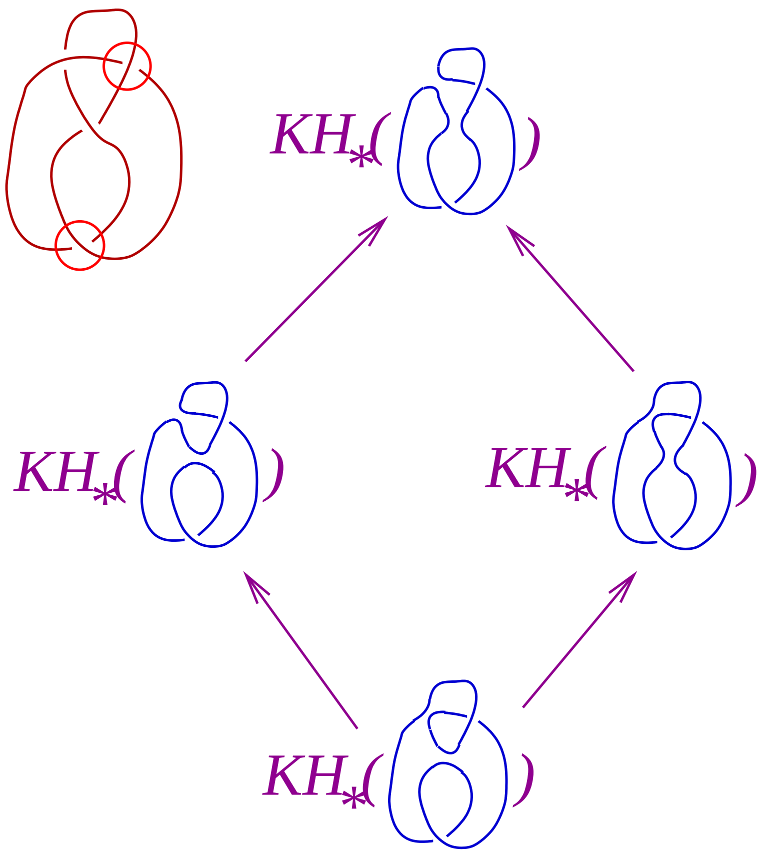


- $A =$ associative R-algebra.
- $P(\Gamma) =$ quiver poset of directed graph Γ.
- $P(\Gamma) \xrightarrow{F_A}$ $\begin{array}{ccc} A\otimes A & & A\otimes A \\ & \nwarrow \quad \nearrow & \\ & A\otimes A\otimes A & \end{array}$
- "homology of A with coefficients in Γ"
  $:= \mathcal{H}_*(P(\Gamma), F_A)$
- **Corollary** [Turner-Wagner]:

$$\mathcal{H}_i(P(n\text{-gon}), F_A) \cong HH_i(A),\ (0 \leq i \leq n-1)$$

($HH_*(A) =$ Hochschild homology)

- Take an N-crossing link diagram D and fix k crossings.
- Resolve each of the remaining crossings as usual.
- Put the resulting 2^{N-k} diagrams on a Boolean lattice $\mathscr{B}$.
- Define a sheaf on $\mathscr{B}$ by taking $KH_*(-)$.



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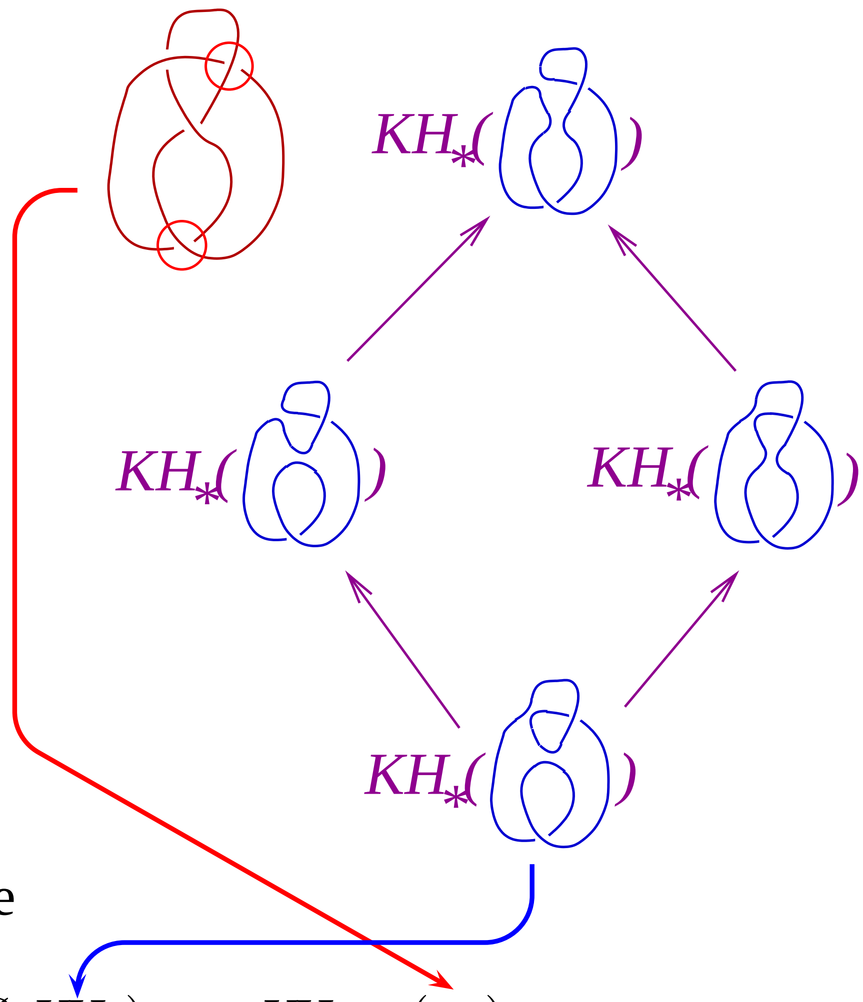

Theorem: There is a spectral sequence

$$E^2_{p,q} = KH_p(\mathscr{B}, KH_q) \Longrightarrow KH_{p+q}(\quad)$$