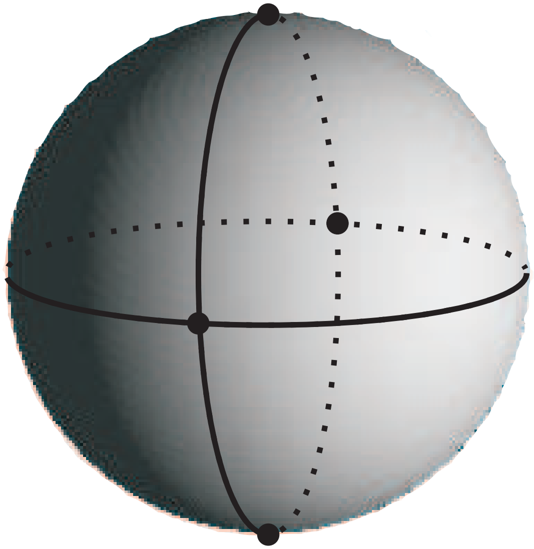
Knots, posets and sheaves

Brent Everitt (York) –joint with Paul Turner (Geneva-Fribourg)

space

$X =$



Euler characteristic:

$$\chi(X) = \sum(-1)^i |X_i|$$

$$(= 2)$$

homology:

$H_*(X;\mathbb{Q}) =$

$\cdots$	$\mathbb{Q}$	0	$\mathbb{Q}$	$\cdots$
	0	1	2	

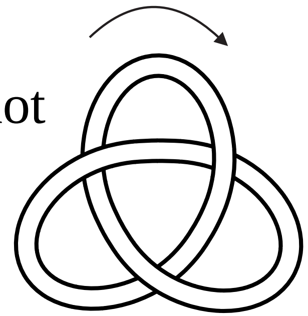
$$\chi = \sum(-1)^i \dim H_i(X)$$

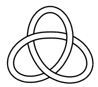
$$(= 2)$$

$$X \xrightarrow{f} Y \rightsquigarrow H_*(X,\mathbb{Q}) \xrightarrow{f_*} H_*(Y,\mathbb{Q})$$

continuous map homomorphism

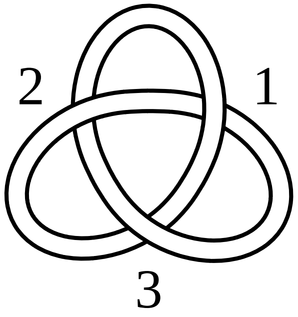
oriented knot

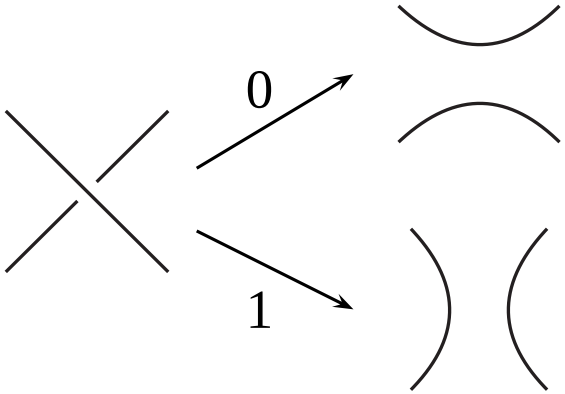
$K =$ 

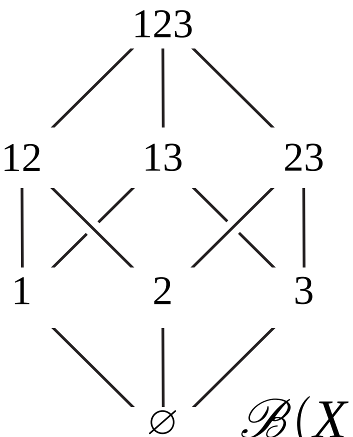
$J\Big($  $\Big) = q^3(-q^6 + q^2 + 1 + q^{-2})$

Jones polynomial

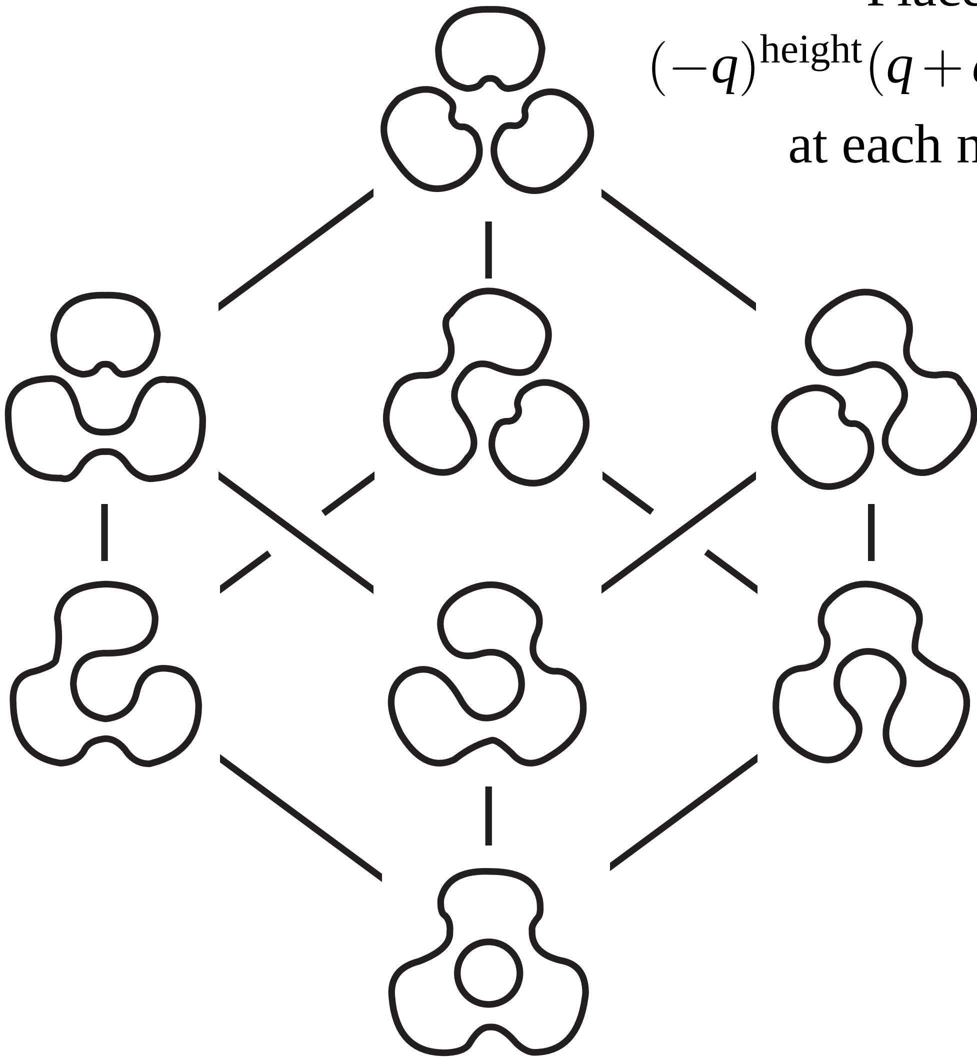
?







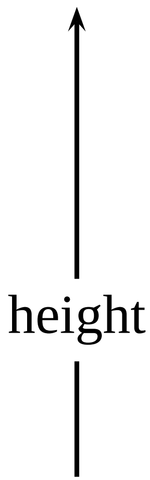
$\mathscr{B}(X), X = \{1,2,3\}$

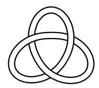
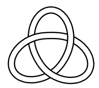


Place

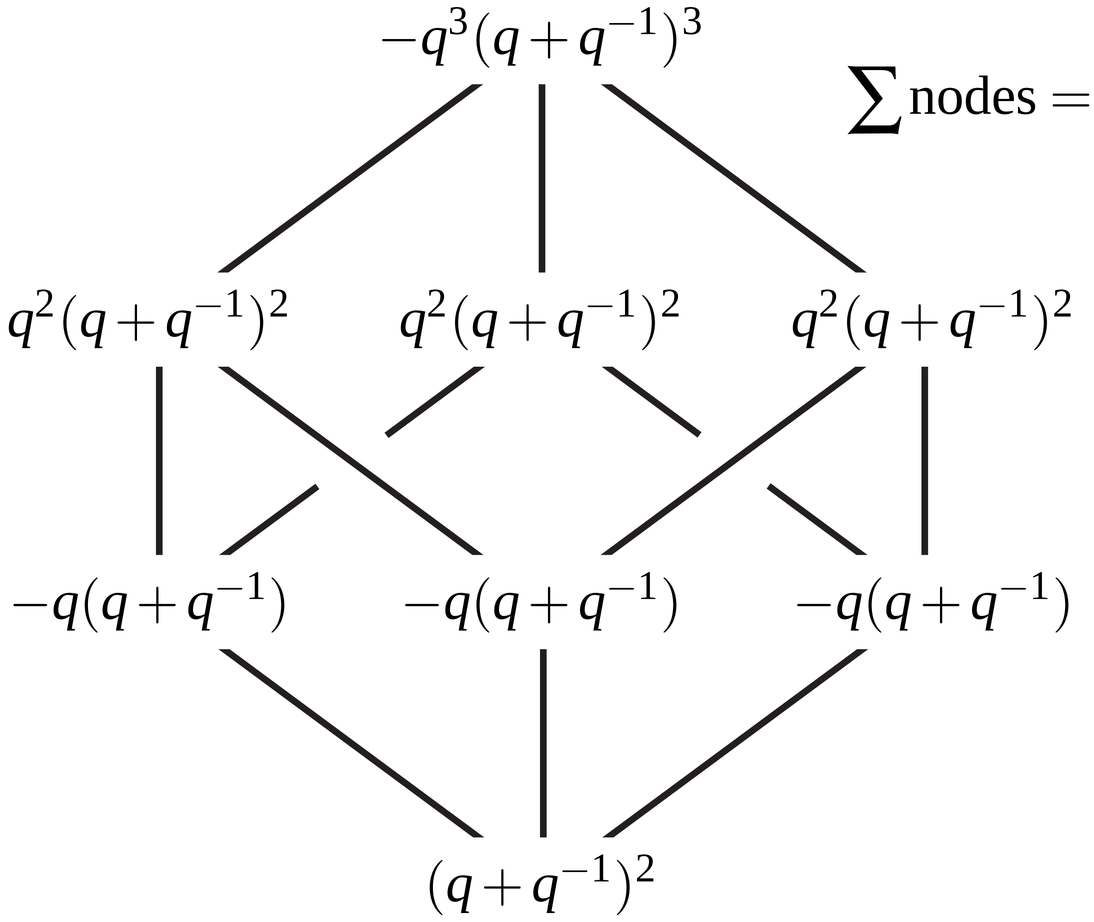
$(-q)^{\text{height}}(q+q^{-1})^{\#\text{circles}}$

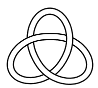
at each node

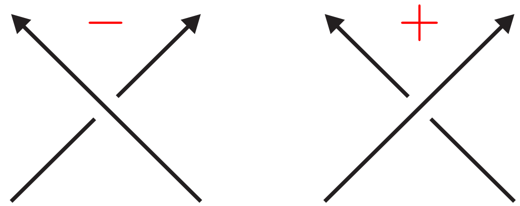


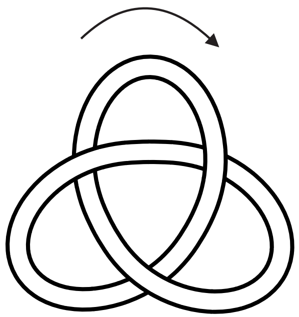
$J\Big($  $\Big) \longleftarrow (-1)^{n_-} q^{n_+ - 2n_-} \Big\langle$  $\Big\rangle$

(Kauffman bracket)



$\sum \text{nodes} = \Big\langle$  $\Big\rangle = -q^6 + q^2 + 1 + q^{-2}$





- $A = \bigoplus A_i =$

$\cdots$	A_{-1}	A_0	A_1	$\cdots$
	-1	0	1	

(A_i = vector spaces over k)

- direct sum $A \oplus B = \bigoplus (A_i \oplus B_i)$

- tensor product $A \otimes B = \bigoplus_{m=-\infty}^{\infty} (A \otimes B)_m$ with $(A \otimes B)_m = \bigoplus_{i+j=m} A_i \otimes B_j$

- $A[\ell] =$

	$A_{-\ell-1}$	$A_{-\ell}$	$A_{\ell+1}$	
	-1	0	1	

$=$

	0	k	0	
	$\ell-1$	ℓ	$\ell+1$	

$\otimes A$

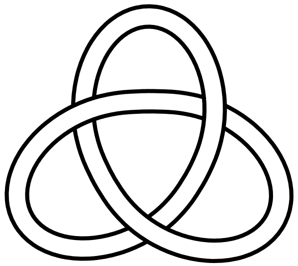
- graded dimension $q\dim A := \sum \dim A_j q^j \in \mathbb{Z}[q, q^{-1}]$

- $q\dim(A \oplus B) = q\dim A + q\dim B$
  $q\dim(A \otimes B) = q\dim A \times q\dim B$
  (e.g.: $q\dim A[\ell] = q^\ell \times q\dim A$)

$$\mathbb{N} = \{0, 1, 2, \ldots\} \quad \underset{\text{de-categorify (dim)}}{\overset{\text{categorify}}{\rightleftarrows}} \quad \text{Spaces}$$

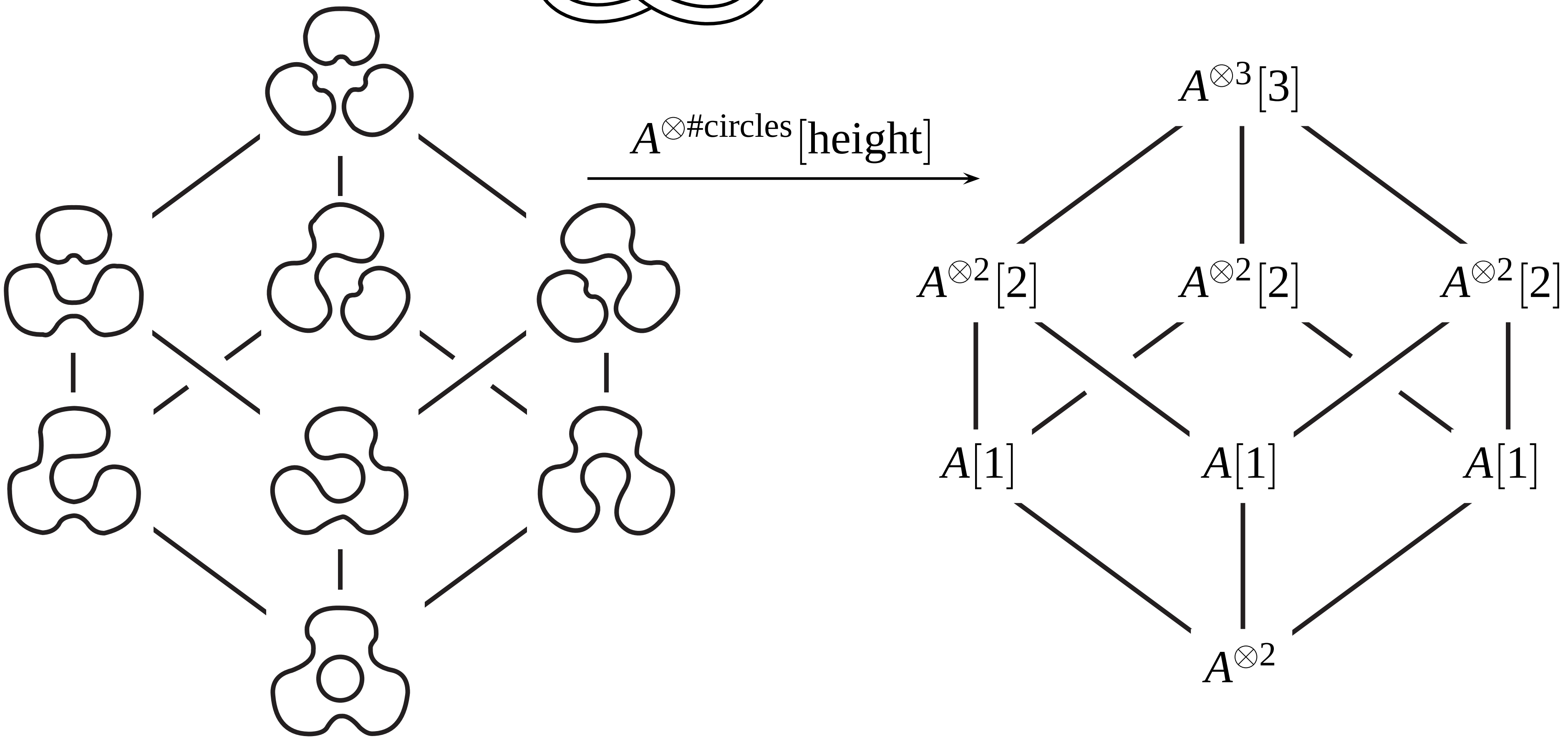
Obj: k-spaces V
Mor: k-linear maps

$$\mathbb{N}[q, q^{-1}] = \{\textstyle\sum_{\mathbb{Z}} n_i q^i\} \quad \underset{\text{de-categorify } (q\text{dim})}{\overset{\text{categorify}}{\rightleftarrows}} \quad \text{Graded Spaces}$$

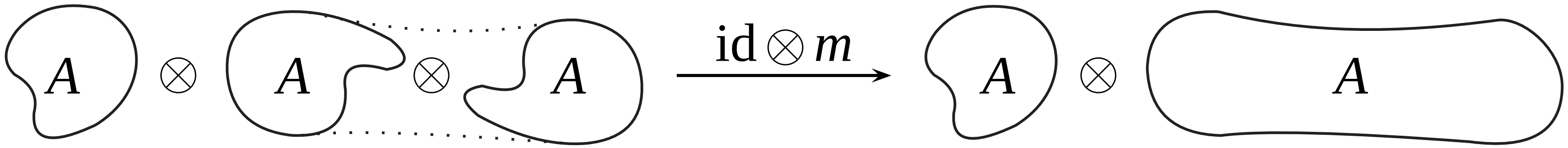


$A = \mathbb{Q} \oplus \mathbb{Q}$

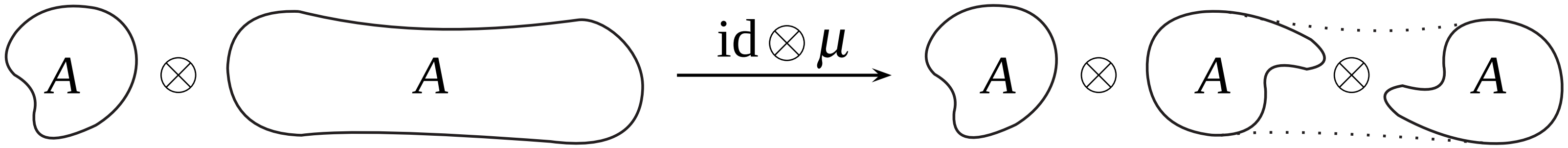
$-1 \quad 1$

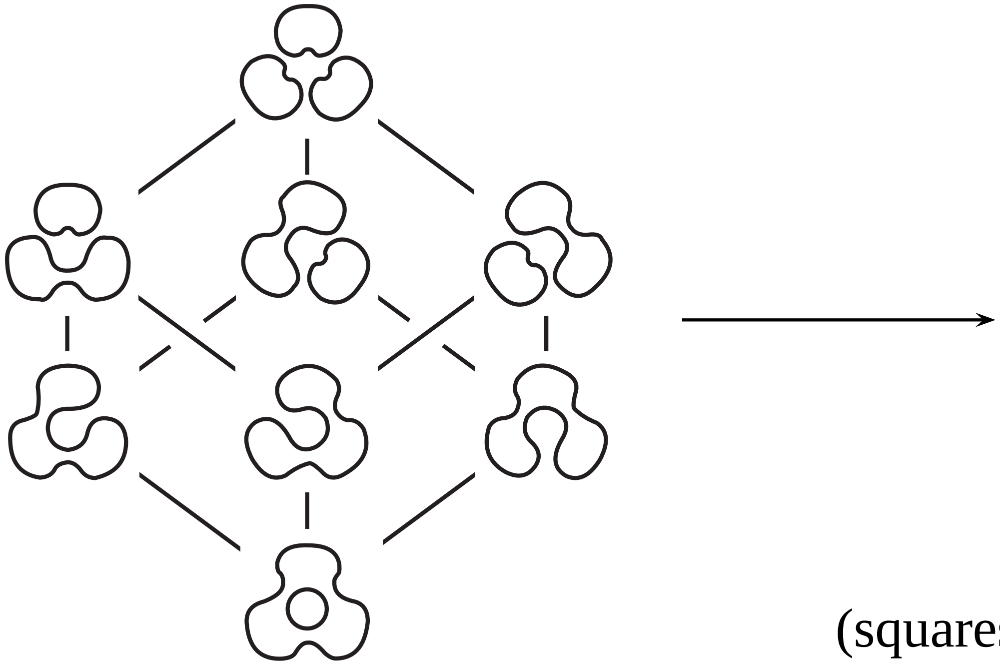


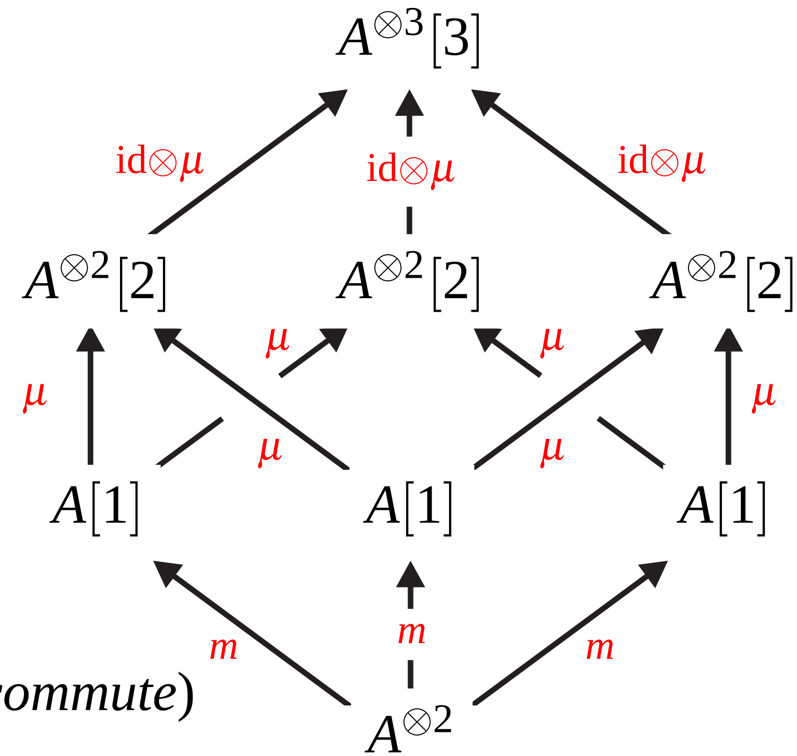
$m : A \otimes A \to A$



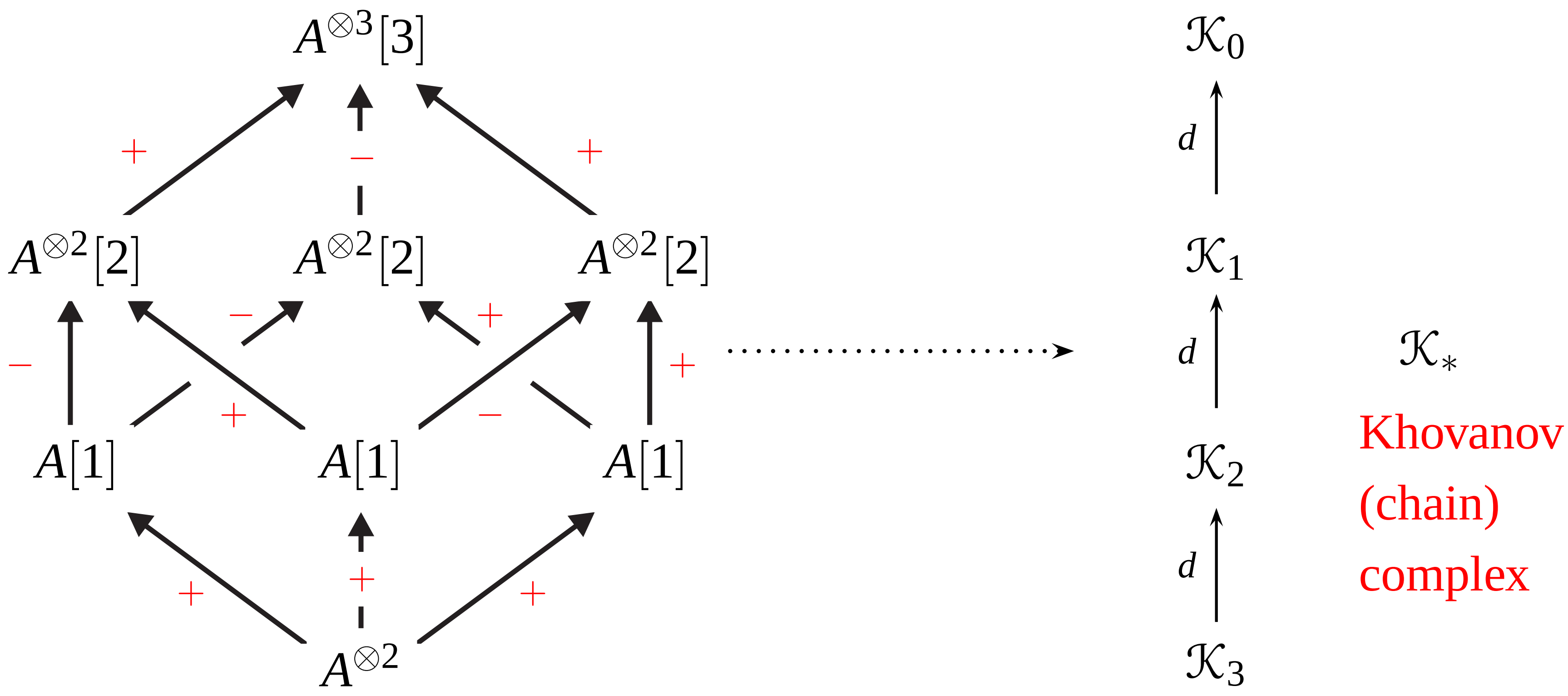
$\mu : A \to A \otimes A$



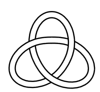


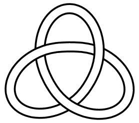


(squares *commute*)

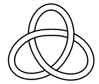


add $\pm$'s to edge maps so squares *anticommute*

Khovanov homology $KH_*\Big($  $,\mathbb{Q}\Big) = H_*(\mathcal{K}_*)$

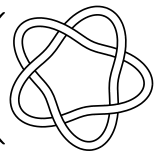
	6	4	2	0	-2	$q\dim$
KH_0	$\mathbb{Q}$					q^6
KH_1			$\mathbb{Q}$			q^2
KH_2						0
KH_3				$\mathbb{Q}$	$\mathbb{Q}$	$1+q^{-2}$

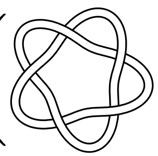
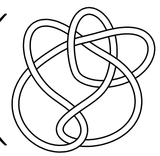
Euler characteristic $\chi(\mathcal{K}_*)$

$= \sum(-1)^i q\dim KH_i\Big($  $, \mathbb{Q}\Big)$

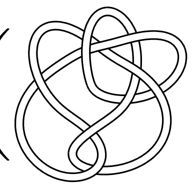
$= q^6 - q^2 - 1 - q^{-2}$

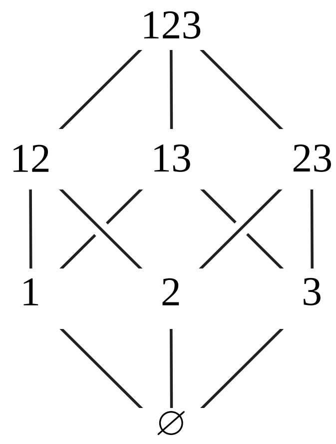
$\mathbb{Q}$						
		$\mathbb{Q}$				
		$\mathbb{Q}$				
				$\mathbb{Q}$		
					$\mathbb{Q}$	$\mathbb{Q}$

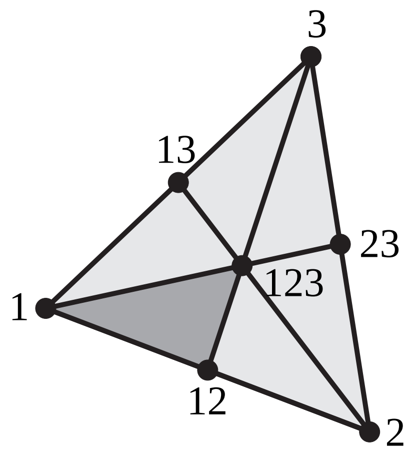
KH_* ()

- minor miracle: KH_* an invariant
- Jones () $=$ Jones ()
- FUNCTORIAL!!

$\mathbb{Q}$							
		$\mathbb{Q}$					
		$\mathbb{Q}$					
			$\mathbb{Q}$	$\mathbb{Q}$			
			$\mathbb{Q}$		$\mathbb{Q}$		
					$\mathbb{Q}\oplus\mathbb{Q}$		
							$\mathbb{Q}$
						$\mathbb{Q}$	$\mathbb{Q}$

KH_* ()





- poset $P \longrightarrow \Delta P$ order (simplicial) complex.

- poset homology $=$ simplicial homology of ΔP

ie: $H_*(P,R) := H_*(\Delta P,R) =$ homology of chain complex

$$C_n(P,R) = \bigoplus_{x_0<\cdots<x_n} R$$

with differential $d : C_n(P,R) \to C_{n-1}(P,R)$

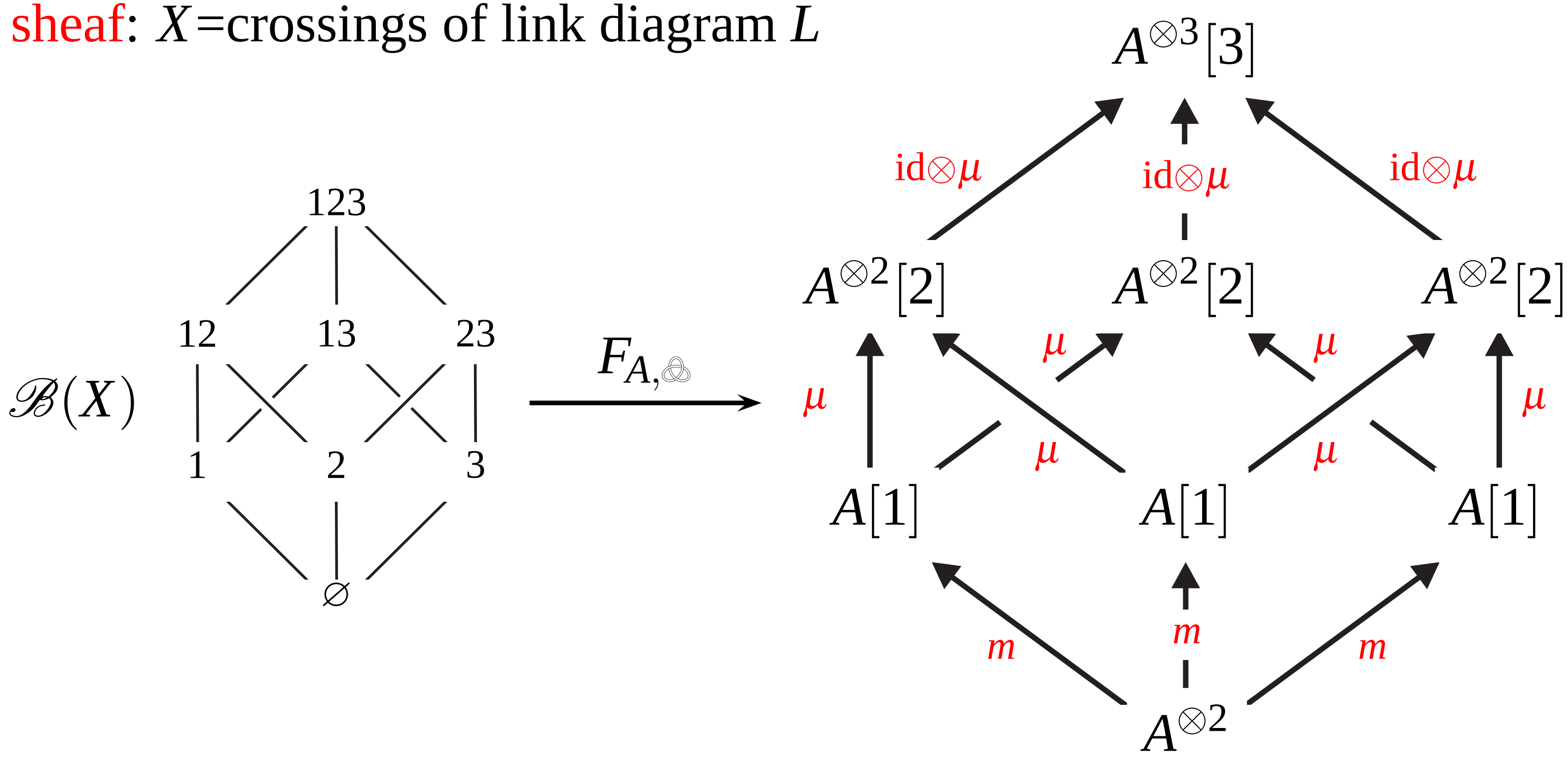
$$\lambda \cdot (x_0 < \cdots < x_n) \overset{d}{\mapsto} \sum_{j=0}^{n} (-1)^j \lambda \cdot (x_0 < \cdots < \widehat{x}_j < \cdots < x_n)$$

- Eg: [Folkman] P finite geometric lattice

$$\widetilde{H}_n(P \setminus \{0,1\},\mathbb{Z}) = \begin{cases} \mathbb{Z}^{|\mu(0,1)|} & n = \text{rk}P - 2, \\ 0 & \text{otherwise.} \end{cases}$$

- $P \xrightarrow{F} R$-**mod** (covariant) functor (= pre-sheaf of R-modules over P)

- Eg: Khovanov sheaf: X=crossings of link diagram L



- $P \xrightarrow{F} R$-**mod** sheaf

- sheaf homology $\mathcal{H}_*(P,F)$ = homology of chain complex

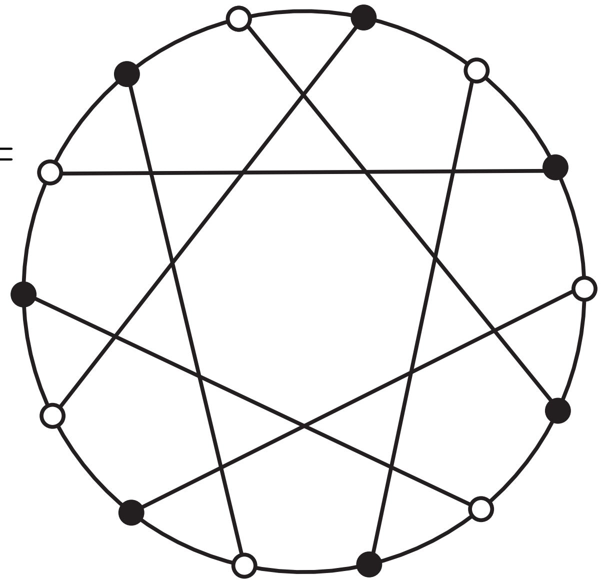
$$S_n(P,F) = \bigoplus_{x_0<\cdots<x_n} F(x_0)$$

with differential $d : S_n(P,F) \to S_{n-1}(P,F)$

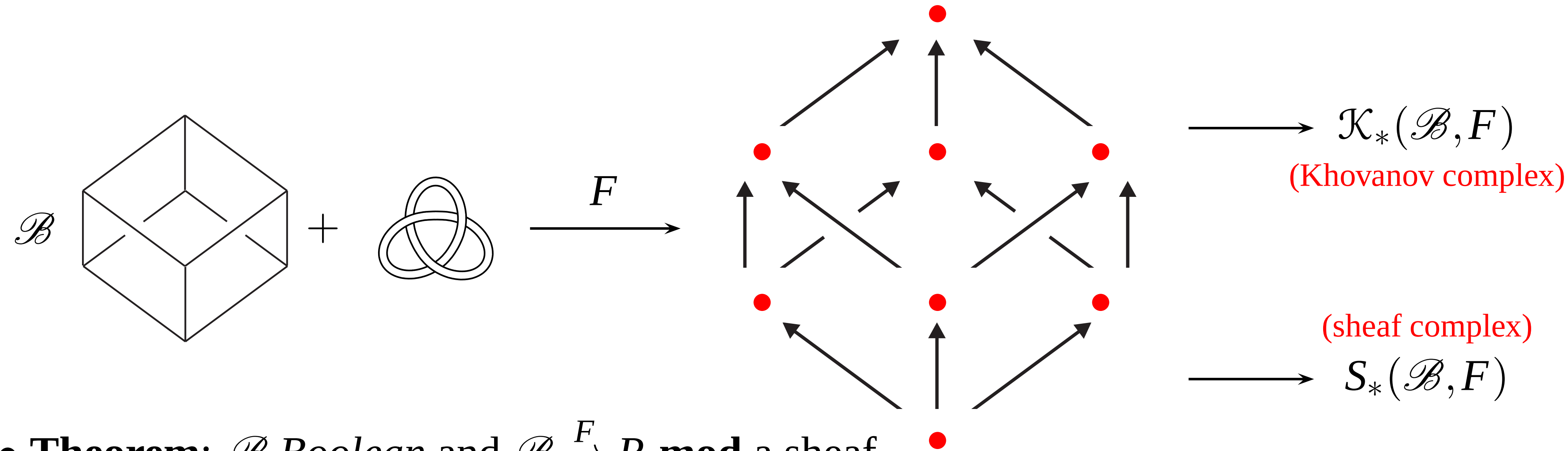
$$\lambda \cdot (x_0 < \cdots < x_n) \overset{d}{\mapsto} F(x_0 < x_1)(\lambda) \cdot (\widehat{x_0} < x_1 < \cdots < x_n)$$
$$+ \sum_{j=1}^{n} (-1)^j \lambda \cdot (x_0 < \cdots < \widehat{x_j} < \cdots < x_n)$$

• sheaf on a building: V = finite dim k-space; P = proper, non-trivial subspaces under $\subseteq$; sheaf $F(U) = U, F(U_1 \subseteq U_2) = U_1 \hookrightarrow U_2$

• E.g.: V 3-dim over $k = \mathbb{F}_2$; $\Delta P =$ (Building of type $A_2(2)$)



• **Theorem** [Lusztig] $\mathcal{H}_n(P,F) = \begin{cases} V & n = 0, \\ 0 & \text{otherwise.} \end{cases}$



• **Theorem**: $\mathscr{B}$ *Boolean* and $\mathscr{B} \xrightarrow{F}$ R-**mod** a sheaf,

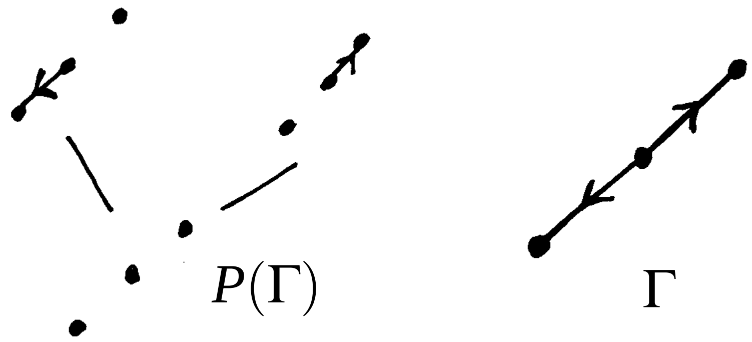
$$KH_*(\mathscr{B},F) \cong \widetilde{\mathcal{H}}_{*-1}(\mathscr{B} \setminus 1, F)$$

• [more generally: one can replace KH_* by a “cellular” homology $H_*^{\text{cell}}(P,F)$ that makes sense for any P:

Theorem: P “cellular” poset and $P \xrightarrow{F}$ R-**mod** a sheaf, then

$$H_*^{\text{cell}}(P,F) \cong \mathcal{H}_*(P,F)$$

many interesting posets turn out to be cellular . . .]



- $A =$ associative R-algebra.

- $P(\Gamma) =$ quiver poset of directed graph Γ.

- $P(\Gamma) \xrightarrow{F_A}$

$$\begin{array}{ccc} A \otimes A & & A \otimes A \\ & \nwarrow \quad \nearrow & \\ & A \otimes A \otimes A & \end{array}$$

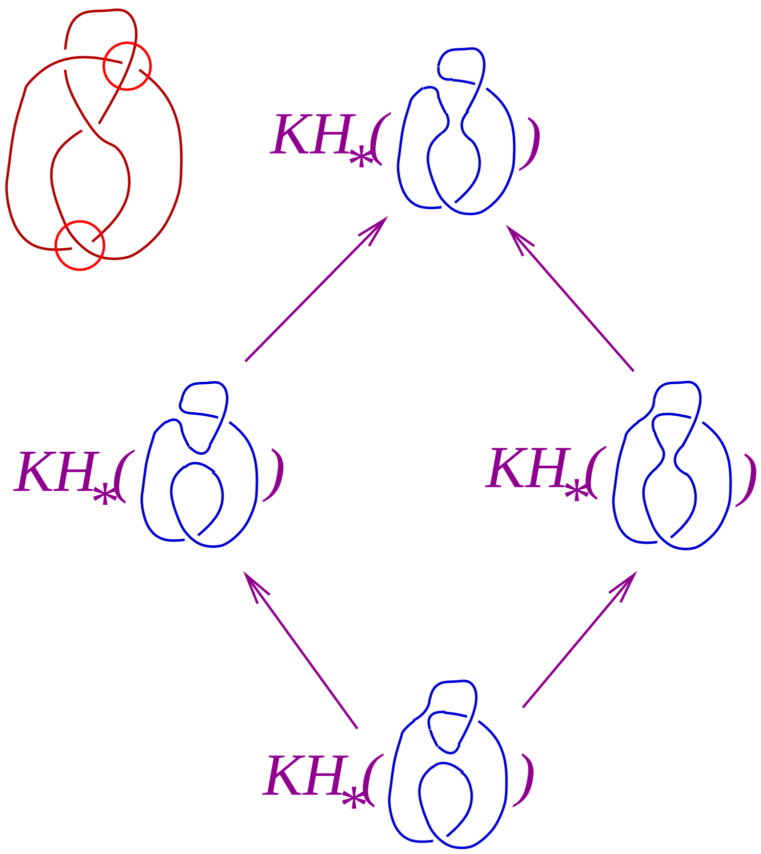
- "homology of A with coefficients in Γ"
  $:= \mathcal{H}_*(P(\Gamma), F_A)$

- **Corollary** [Turner-Wagner]:

$$\mathcal{H}_i(P(n\text{-gon}), F_A) \cong HH_i(A), \ (0 \leq i \leq n-1)$$

($HH_*(A) =$ Hochschild homology)

- Take an N-crossing link diagram D and fix k crossings.
- Resolve each of the remaining crossings as usual.
- Put the resulting 2^{N-k} diagrams on a Boolean lattice $\mathscr{B}$.
- Define a sheaf on $\mathscr{B}$ by taking $KH_*(-)$.



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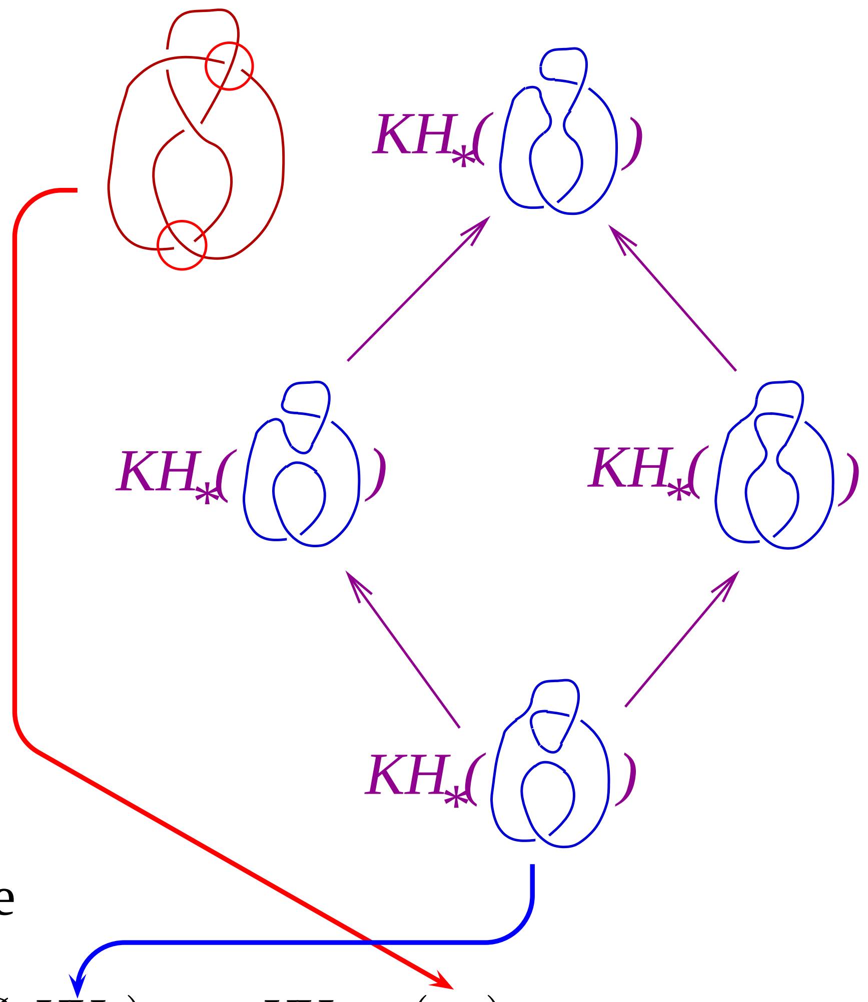

Theorem: There is a spectral sequence

$$E^2_{p,q} = KH_p(\mathscr{B}, KH_q) \Longrightarrow KH_{p+q}(\quad)$$