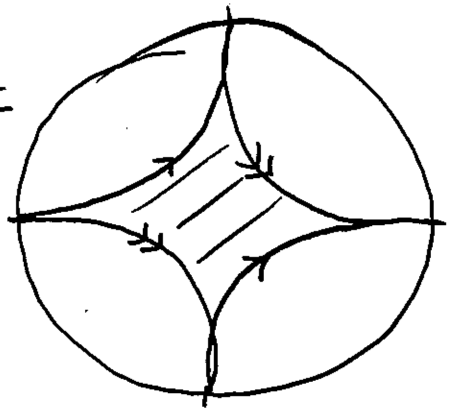
S. Tschantz)

$X = n$-dim. hyperbolic manifold

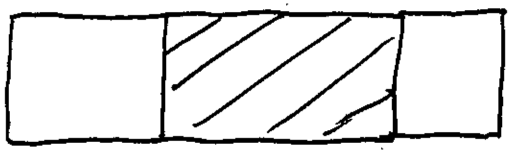
$X \begin{cases} \text{complete} \\ \text{conn.} \\ \text{orientable} \\ \text{w/o } \partial \\ \text{finite volume} \end{cases}$

Eg: $X =$  $\approx$ 

torus $\setminus$ pt.

$X = H^n/G \begin{cases} G \curvearrowright H^n \text{ prop. disc.} \\ \text{freely} \\ \text{by isometries} \end{cases} \Longleftrightarrow \begin{cases} G \subset \mathrm{Isom}\, H^n \\ \text{discrete} \\ \text{torsion free} \end{cases}$

<u>to construct $\overbrace{X}^{\text{such}}$</u>: construct G



2. Sociology

$\mathcal{H}(n) = \{n\text{-dim. hyp. mflds}/\text{isometry}\}$

$\mathrm{vol}_n : \mathcal{H}(n) \to \mathbb{R}^{>0}$

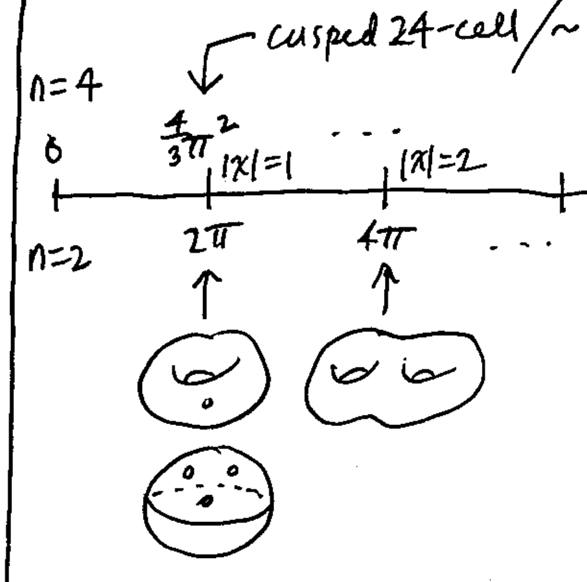
image:

well-ordered $\begin{cases} n \neq 3 & \text{discrete} \\ n = 3 & \text{type } \omega^\omega \end{cases}$

pre-image: $|\mathrm{vol}_n^{-1}(x)| < \infty$, $n > 2$.

Siegel problem: find minimal volume (fixed n)

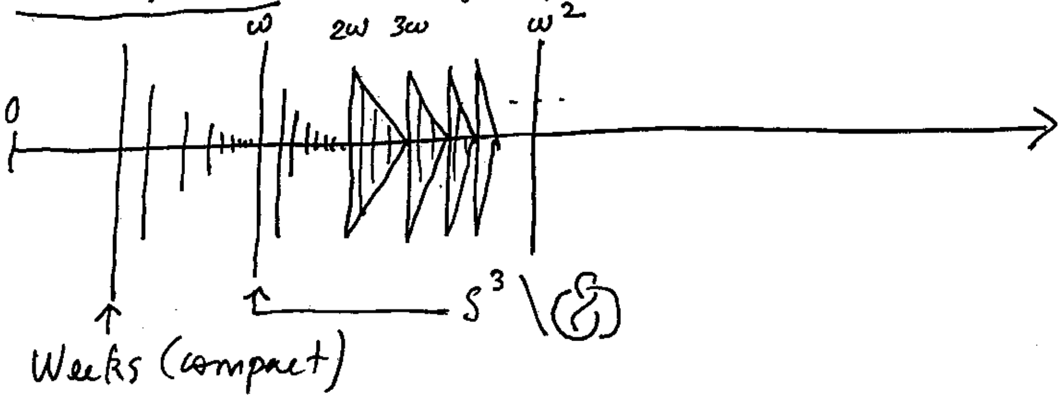
$\mathcal{H}(\text{even})/\mathrm{vol}$:

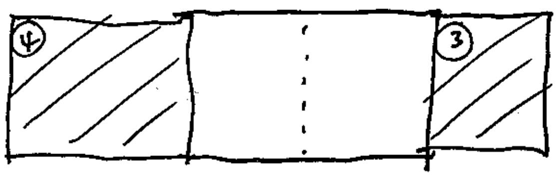


$\mathrm{vol}(M) = \kappa_n \cdot \chi(M)$

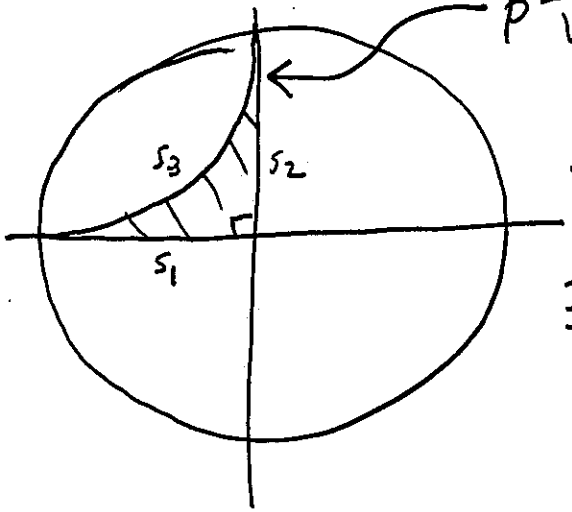
M compact $\Rightarrow \chi(M) \in 2\mathbb{Z}$

$\mathcal{H}(3)/\mathrm{vol}$: (fib of ω^2)





3. toy example



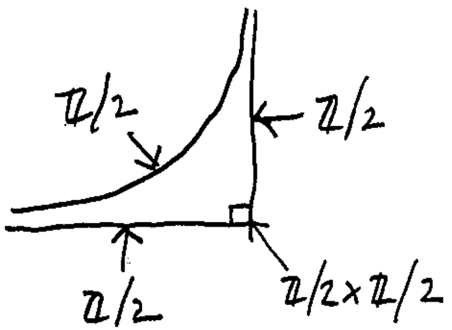
$H = \langle \text{reflections } s_i \rangle$

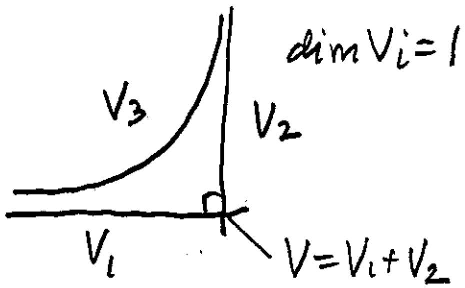
$\cong \langle s_i \mid s_i^2 = (s_1 s_2)^2 = 1 \rangle$

find: $H \twoheadrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2$ with torsion free kernel

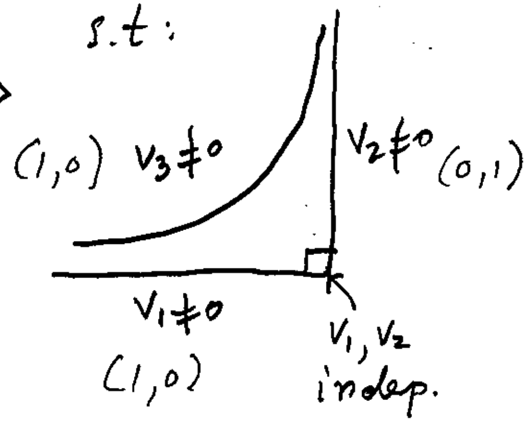
homom: any map $\{s_i\} \to \mathbb{Z}/2 \times \mathbb{Z}/2$ gives one

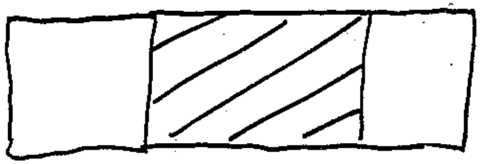
torsion-free kernel: $\underbrace{\text{finite subgps} \subseteq H}_{\text{stabilizers}} \to$ copies of themselves in $\underbrace{\mathbb{Z}/2 \times \mathbb{Z}/2}_{= V \ (2\text{-dim}/\mathbb{F}_2)}$ $\Rightarrow$ label sides by $v \in V$ s.t.:





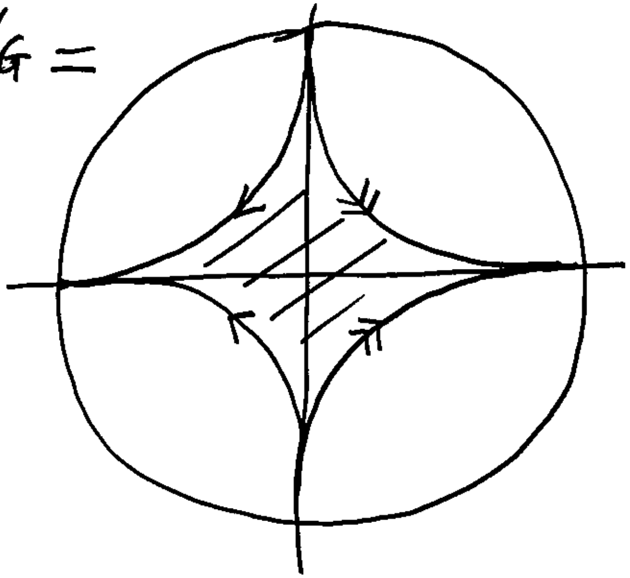
$\dim V_i = 1$



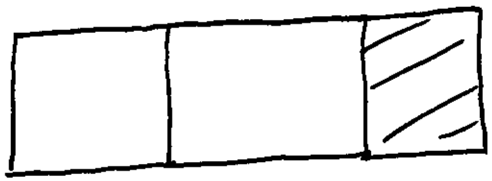


$G = \text{kernel}$

$H^2/G =$



$\approx S^2 \setminus 3\,\text{pts}.$



4. In general (sketch)

- L = (the) odd Lorentzian lattice rank $n+1$

$\operatorname{Aut} L \circlearrowright L \otimes \mathbb{R}$

preserves sphere radius i / scalars $\approx H^n$

action H^n with volume $(2^{n/2} \pm 1) \prod^{n/2} B_{2k} \cdot \frac{\pi^{n/2}}{n!}$

(n even) $\quad (n=2: \frac{\pi}{4})$

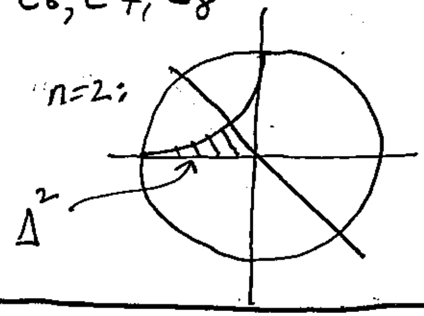
- $2 \le n \le 8$: $\operatorname{Aut} L$ a reflection gp. with region Δ^n

finite subgp. $W \subset \operatorname{Aut} L$

$\llcorner$ Weyl gp. $A_1, A_1 \times A_2, \underset{SL_5}{A_4}, \underset{SO_{10}}{D_5}, E_6, E_7, \underset{\text{order} \approx 10^9}{E_8}$

$P^n = \bigcup$ W-images of Δ^n

$n=2$:



- $P^n \circlearrowleft W \circlearrowright$ representation V

faces $\overset{1-1}{\longleftrightarrow}$ cosets $\overset{1-1}{\longleftrightarrow}$ orbit in root lattice / mod 2

- $M = H^n / G$

$n=2$: $\chi = -1$

$n=4$: $\chi = 1$

$n=6$: $\chi = -8$

$M' = M /$ finite gp. autos $\quad \chi = -1$