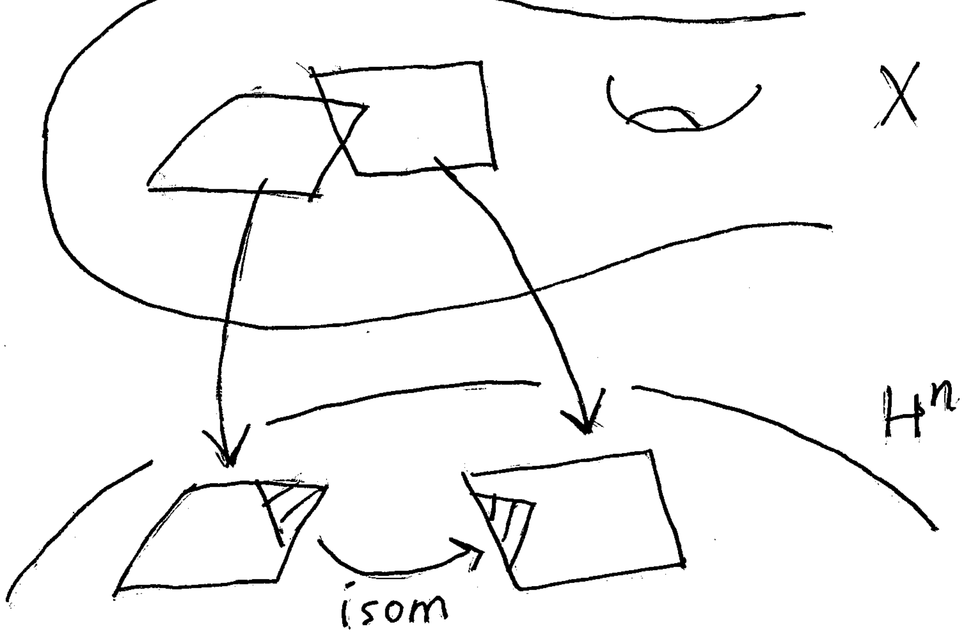
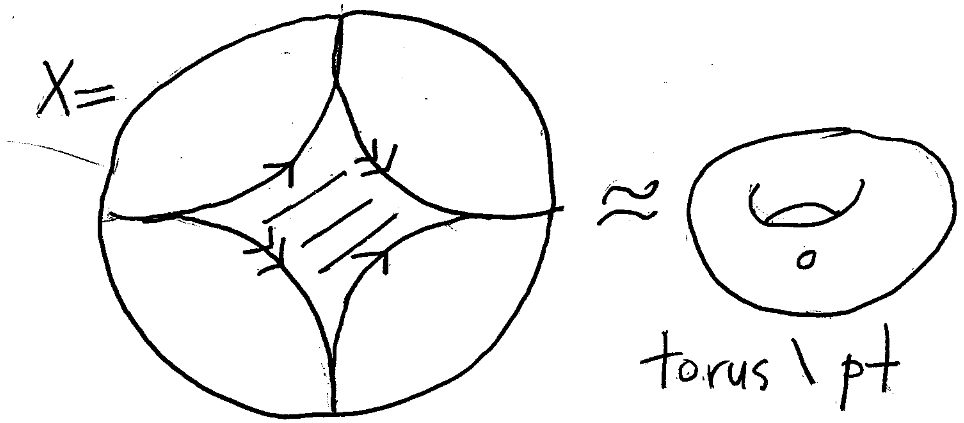
Siegel's problem (joint with J. Ratcliffe / S. Tschantz)

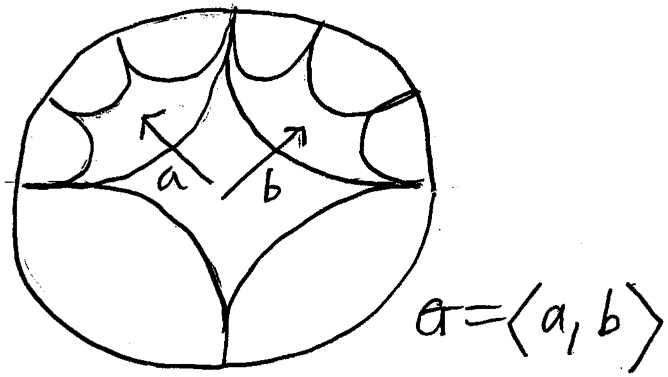
$X =$ n-dim hyperbolic mfld



Eg: $X =$



$$X = H^n/G \left\{\begin{array}{l} G \curvearrowright H^n \text{ prop. disc.} \\ \text{by isometries} \\ \text{freely} \end{array}\right. \iff \left\{\begin{array}{l} G \subset \mathrm{Isom}\, H^n \\ \text{discrete} \\ \text{torsionfree} \end{array}\right.$$



$G = \langle a, b \rangle$

to construct such X: construct G

2. Sociology

$\mathcal{H}(n) = \{n\text{-dim } \overset{\text{orientable}}{\vee} \text{ hyp. mflds}/\text{isometry}\}$

$\mathrm{vol}_n : \mathcal{H}(n) \to \mathbb{R}^{\geq 0} \cup \{\infty\}$

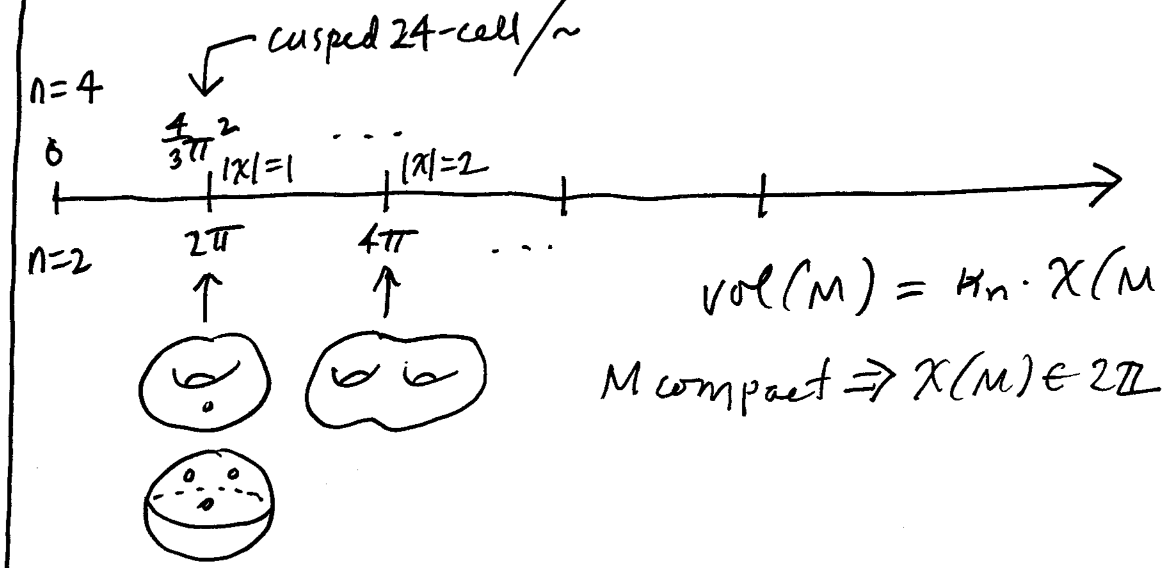
image:

well-ordered $\begin{cases} n \neq 3 & \text{discrete} \\ n = 3 & \text{type } \omega^\omega \end{cases}$

pre-image : $|\mathrm{vol}_n^{-1}(x)| < \infty, \ n > 2.$

Siegel problem : find minimal volume (fixed n)

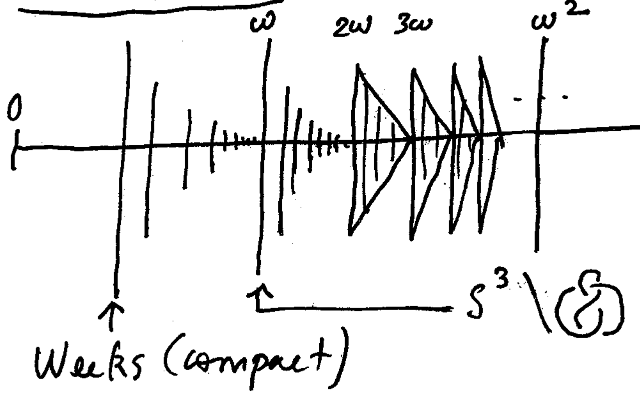
$\mathcal{H}$(even)/vol:

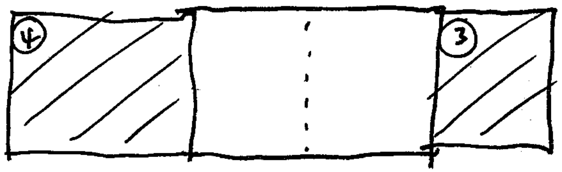


$\mathrm{vol}(M) = \kappa_n \cdot \chi(M)$

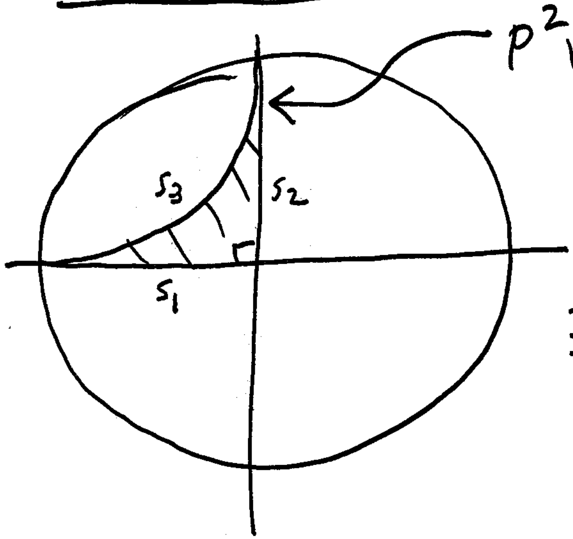
M compact $\Rightarrow \chi(M) \in 2\mathbb{Z}$

$\mathcal{H}(3)$/vol : (first ω^2)





3. toy example



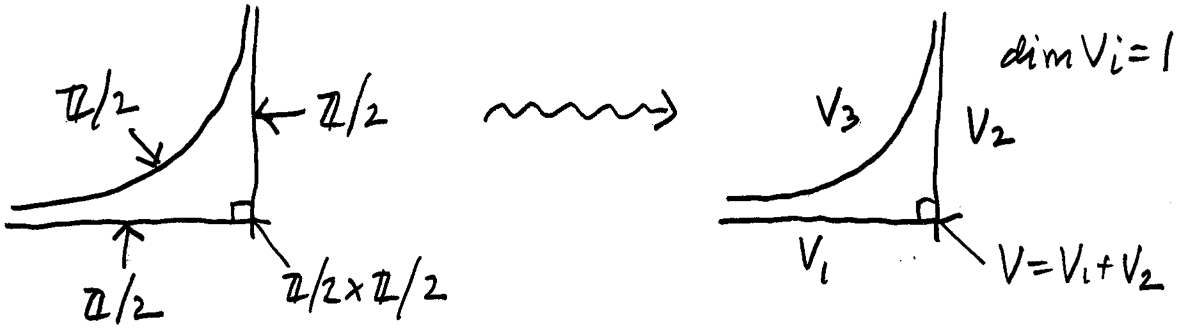
$H = \langle$ reflections $s_i \rangle$

$\cong \langle s_i \mid s_i^2 = (s_1 s_2)^2 = 1 \rangle$

find: $H \twoheadrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2$ with torsion free kernel

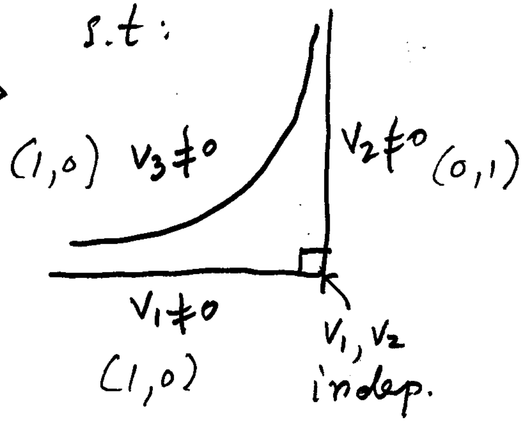
homom: any map $\{s_i\} \to \mathbb{Z}/2 \times \mathbb{Z}/2$ gives one

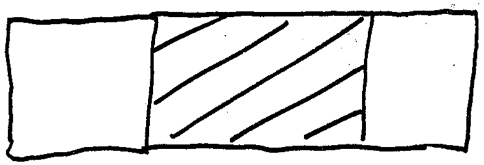
torsion-free kernel: $\underbrace{\text{finite subgps} \subseteq H}_{\text{stabilizers}} \to$ copies of themselves in $\underbrace{\mathbb{Z}/2 \times \mathbb{Z}/2}_{= V \text{ (2-dim}/\mathbb{F}_2)}$



$\dim V_i = 1$

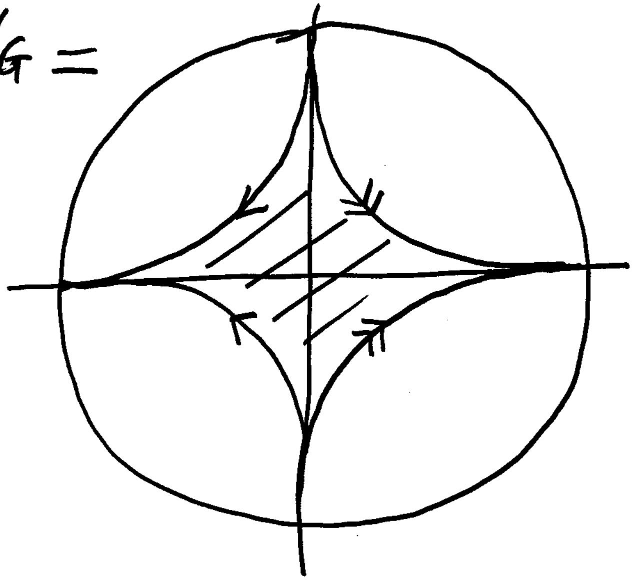
$\Rightarrow$ label sides by $v \in V$ s.t.:



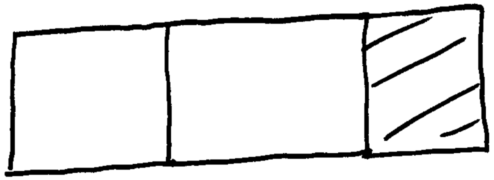


$G =$ kernel

$H^2/G =$



$\approx S^2 \setminus 3$ pts.



4. In general (sketch)

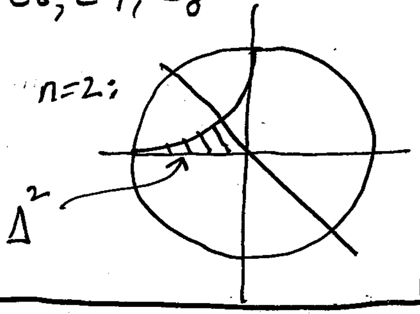
- $L =$ (the) odd Lorentzian lattice rank $n+1$

$\mathrm{Aut}\, L \circlearrowleft L \otimes \mathbb{R}$
preserves sphere radius i / scalars $\approx H^n$
acts on H^n with volume $(2^{n/2} \pm 1) \prod^{n/2} B_{2k} \cdot \frac{\pi^{n/2}}{n!}$
(n even) $\left(n=2: \frac{\pi}{4}\right)$

- $2 \le n \le 8$: $\mathrm{Aut}\, L$ a reflection gp. with region Δ^n

finite subgp. $W \subset \mathrm{Aut}\, L$
$\llcorner$ Weyl gp. $A_1, A_1 \times A_2, \underset{SL_5}{A_4}, \underset{SO_{10}}{D_5}, E_6, E_7, \underset{\text{order} \approx 10^9}{E_8}$

$P^n = \bigcup$ W-images of Δ^n



- $P^n \circlearrowleft W \circlearrowright$ representation V

faces $\overset{1-1}{\longleftrightarrow}$ cosets $\overset{1-1}{\longleftrightarrow}$ orbit in root lattice / mod 2

- $M = H^n / G$

$n=2$: $\chi = -1$

$n=4$: $\chi = 1$

$n=6$: $\chi = -8$

$M' = M /$ finite gp. autos $\quad \chi = -1$