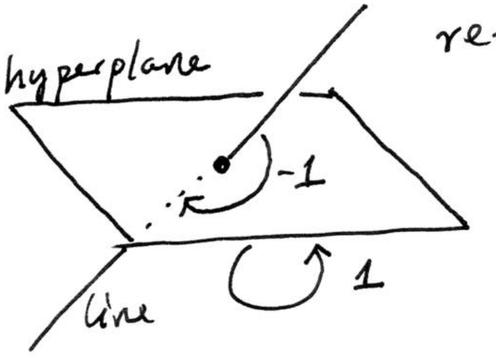
① Reflection groups:

$V =$ (finite dim.) $\mathbb{R}$-space

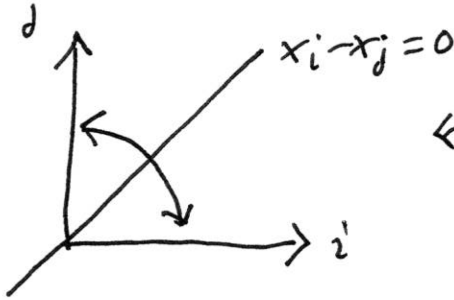
reflection
$s : V \to V$



$W = \langle$ finitely many refs. $\rangle \subset GL(V)$

Eg: V Euclidean

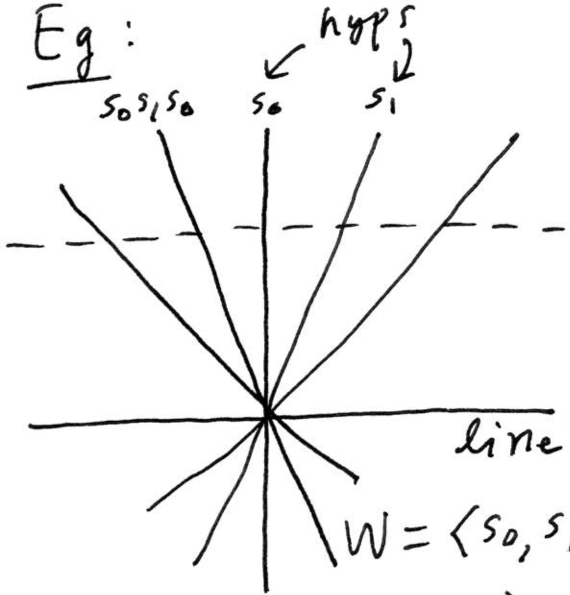
$W = \langle$ refs in $x_i - x_j = 0 \rangle$



$\leftrightsquigarrow$ $S_n \circlearrowright V$

(Symmetric gp. as reflection gp.)

Eg:



$W = \langle s_0, s_1 \rangle$

(∞-dihedral gp.)

Eg: $G =$ (reductive) algebraic gp. ($GL_n k$)

$T \subset G$ maximal torus (diagonal matrices)

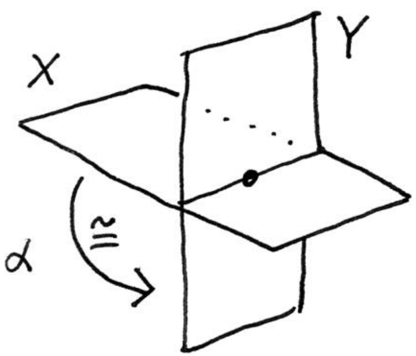
$W = \mathrm{Aut}(T) \circlearrowright$ (characters of T) $\otimes \mathbb{R}$

(permutation matrices $\cong S_n$ $\circlearrowright$ n-dim. V)

Weyl gp.

(2) Reflection monoids:

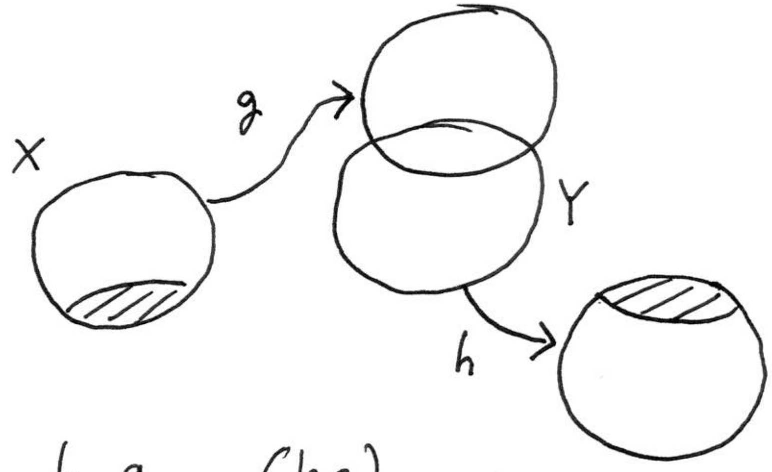
partial linear isomorphism:



X, Y subspaces

$\alpha = g|_X \; (:= g_X)$

$g \in GL(V)$



$h_Y g_X = (hg)_{X \cap g^{-1}Y}$

system $\mathcal{S}$ of subspaces

for (ref.) gp. W: $\begin{cases} V \in \mathcal{S} \\ X, Y \in \mathcal{S} \Rightarrow X \cap Y \in \mathcal{S} \\ W\mathcal{S} = \mathcal{S} \end{cases}$

reflection monoid:

$M(W, \mathcal{S}) = \{ g_X \mid g \in W, X \in \mathcal{S} \}$

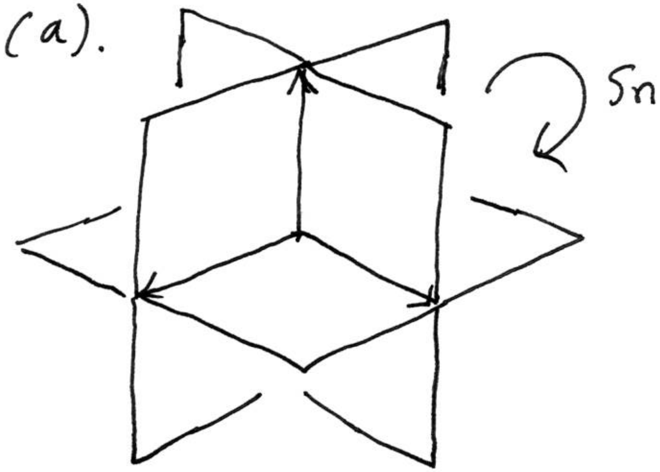
ref. gp. system

$= \left\langle s_X \;\middle|\; \begin{array}{l} s \text{ ref. in } W \\ X \in \mathcal{S} \end{array} \right\rangle$

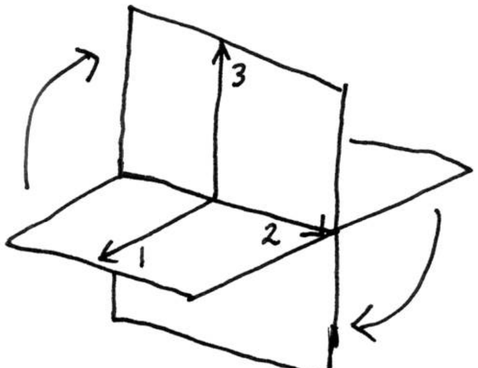
partial ref.

③ Eg's:

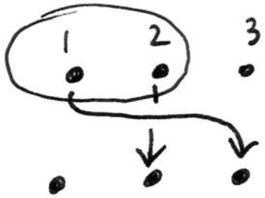
(a).



$\mathcal{S}$ = all $\cap$'s of



$\leftrightsquigarrow$



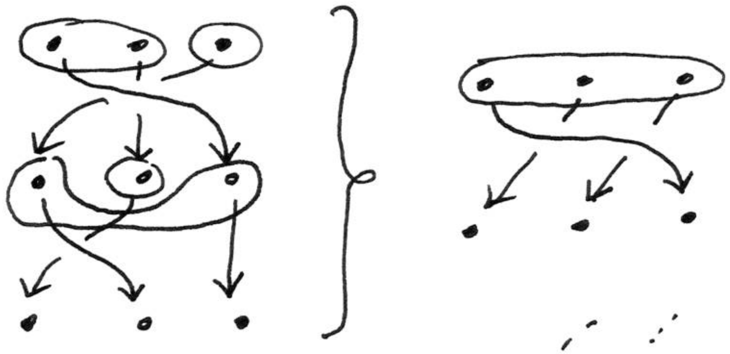
partial isom.

partial perm.

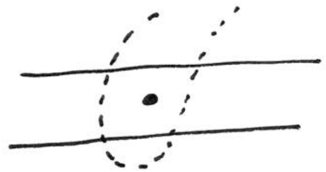
$M(S_n, \mathcal{S}) \cong$ symmetric inverse monoid

(b). $S_n \curvearrowright V$

$\mathcal{S}$ = all $\cap$'s of the $x_i - x_j = 0$
(braid arrangement)



uniform block perms.



$\vee$ of partitions

$M(S_n, \mathcal{S}) \cong$ monoid uniform block perms.

In general:

$|M(W, \mathcal{S})| = \sum$ indices of parabolic subgps. of W

Weyl gp.

$\cap$'s of ref. hyps.

4 Presentations: $W \curvearrowright \mathcal{S}$

(a). (W, S) Coxeter system:

$$W = \langle s \in S \mid (s_\alpha s_\beta)^{m_{\alpha\beta}} = 1 \rangle$$

$$(m_{\alpha\alpha} = 1)$$

(b). $\mathcal{S}$ $\vee$-semilattice under $\supseteq$

$XY := X \vee Y$
- assoc.
- comm. — rels
- $X^2 = X$ — rels

comm. monoid idempotents

Eg: <u>Face monoids of polytopes</u>

facets of polytopes (gens) ; $\mathcal{S}$ = all $\vee$'s

Eg: <u>geometric lattices</u>

$\mathcal{A}$ = hyp. arrangement (gens) ; $\mathcal{S}$ = all $\vee$'s

generators A <u>independent</u>: $\bigvee A' < \bigvee A$ for all $A' \subset A$

rels: $\underbrace{a_1 \cdots a_k}_{\text{indep.}} = a_1 \cdots a_k b$, $\quad b \leq \bigvee a_i$

$\underbrace{\hat{a}_1 \cdots a_k}_{\text{min. dep.}} = \cdots = a_1 \cdots \hat{a}_k$ rels

<u>Eg</u>: braid arrangement

$x_i - x_j = 0$

rels: