Partial mirror symmetry: reflection monoids

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Symmetric

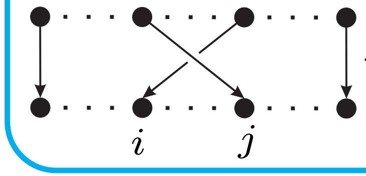
$\boxed{\mathfrak{S}_n}$

group

Permutation groups

bijections $X \to X$

$X = \{1, 2, \ldots, n\}$



$\rightsquigarrow (i, j)$

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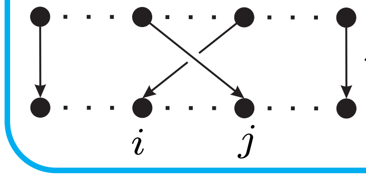

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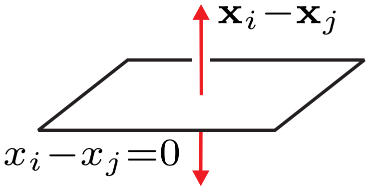
Reflection groups

V Euclidean space

basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$



$\langle$reflection $s_{ij}\rangle \subset \mathrm{GL}(V)$

$s_{ij} \rightsquigarrow (i, j)$

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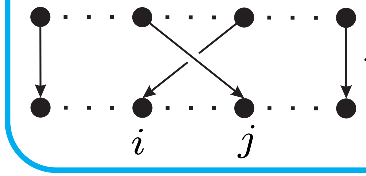

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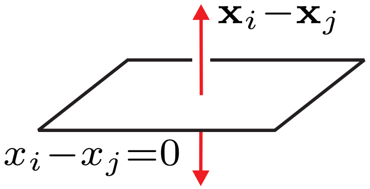


$\langle \text{reflection } s_{ij} \rangle \subset \text{GL}(V)$

$s_{ij} \rightsquigarrow (i, j)$

Weyl groups

$\mathbb{G} = \text{GL}_n(\mathbb{F})$

$T = \text{D}^*_n(\mathbb{F}) \subset \text{GL}_n(\mathbb{F})$ torus

$W = N_{\mathbb{G}}(T)/T$ permutation matrices

$$\begin{array}{c|ccccccccccc|}
 & & & & i & & & & j & & & \\
 & 1 & & & & & & & & & & \\
 & & \ddots & & & & & & & & & \\
 & & & 1 & & & & & & & & \\
i & & & & 0 & & & & 1 & & & \\
 & & & & & 1 & & & & & & \\
 & & & & & & \ddots & & & & & \\
 & & & & & & & 1 & & & & \\
j & & & & 1 & & & & 0 & & & \\
 & & & & & & & & & 1 & & \\
 & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & 1
\end{array} \rightsquigarrow (i, j)$$

Symmetric

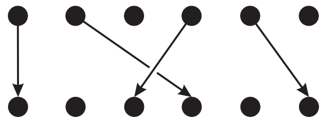
$$\boxed{\mathscr{I}_n}$$

inverse monoid

Partial permutations

bijections $X \supset Y \to Y' \subset X$

$$X = \{1, 2, \dots, n\}$$



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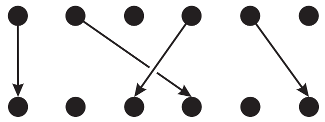


Symmetric

$\mathscr{I}_n$

inverse monoid

Renner monoids

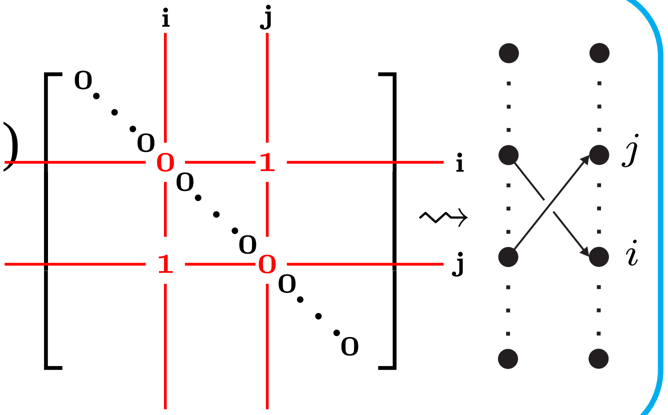
$\mathbb{G} = \mathrm{GL}_n(\mathbb{F}) \subset \mathrm{M}_n(\mathbb{F}) = \mathbb{M}$

$T = \mathrm{D}_n^*(\mathbb{F}) \subset \mathrm{D}_n(\mathbb{F}) = \overline{T}$ (Zariski closure)

$W = N_{\mathbb{G}}(T)/T \subset \overline{N_{\mathbb{G}}(T)}/T$

partial permutation matrices

=Rook monoid



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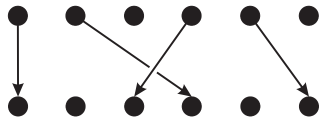

Reflection monoids

?

Symmetric

$$\mathscr{I}_n$$

inverse monoid

Renner monoids

$$\mathbb{G} = \mathrm{GL}_n(\mathbb{F}) \subset \mathrm{M}_n(\mathbb{F}) = \mathbb{M}$$

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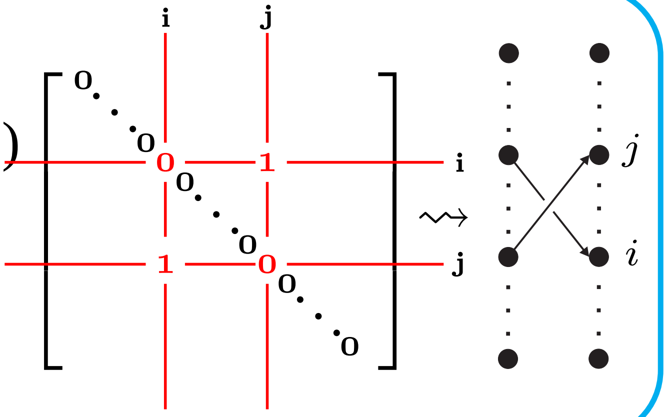

Reflection groups

- $\mathbb{F}$ field, V space over $\mathbb{F}$, reflection: $\mathbb{F} \oplus \mathbb{F} \oplus \cdots \oplus \mathbb{F}$

$$\text{ie: } V \xrightarrow{s} V \in \mathrm{GL}(V) \begin{cases} \text{order} \in \mathbb{Z}^{>1}, \\ \text{fixes hyperplane}, \\ \text{semisimple}. \end{cases}$$

- reflection group $:= \langle \text{reflections } s \in S \rangle \subset \mathrm{GL}(V)$.

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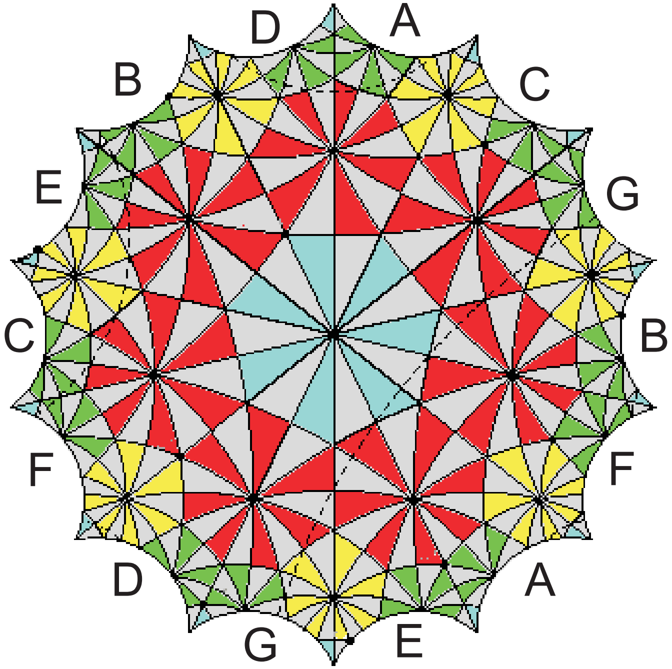
Reflection groups

- $\mathbb{F} = \mathbb{F}_q$ (q odd), Q quadratic form on V, $O(V, Q) =$ orthogonal group. ($O(V, Q) = O_n^{\pm}(q)$, n even, or $O_n^{\circ}(q)$, n odd)

- $\mathbb{F} = \mathbb{C}$: Klein's quartic $X^3Y + Y^3Z + Z^3X = 0 \subset \mathbb{CP}^2$



$$\begin{tikzcd} G_{24} = \langle s_1, s_2, s_3 \rangle \subset \mathrm{GL}_3\mathbb{C} \arrow[d, "\text{mod}"', "\text{scalars}"] \\ \mathrm{PSL}_2\mathbb{F}_7 \end{tikzcd}$$

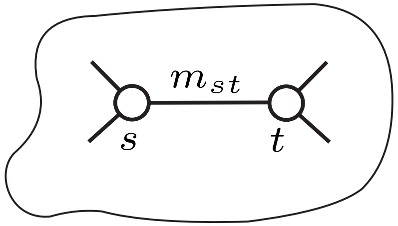
Reflection groups

- $\mathbb{F} = \mathbb{R}$: Coxeter groups:

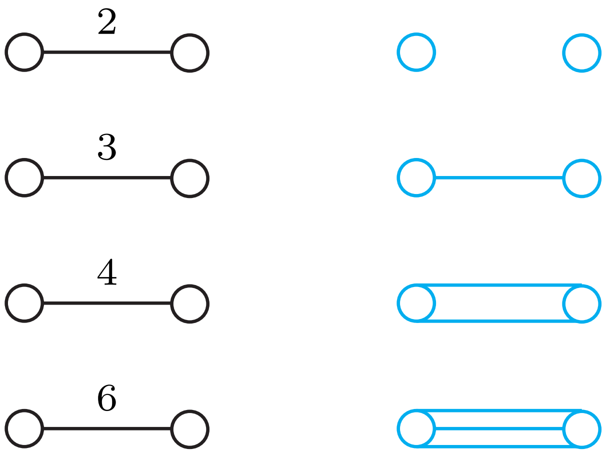
$$(W, S) = \langle s \in S \,|\, (st)^{m_{st}} = 1 \rangle \qquad m_{st} \in \mathbb{Z}^{\geq 1} \cup \{\infty\}$$

$$m_{st} = 1 \Leftrightarrow s = t$$

Coxeter symbol Γ:

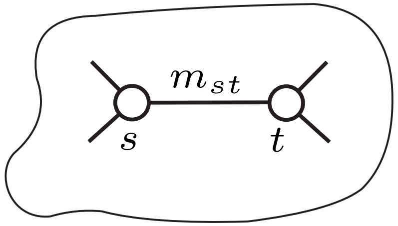


Coxeter Dynkin



Reflection groups

Coxeter groups are real reflection groups!



$$V := \langle \mathbf{v}_s \,|\, s \in \Gamma \rangle_{\mathbb{R}}$$

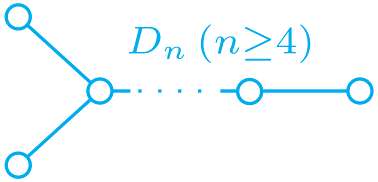
$$B(\mathbf{v}_s, \mathbf{v}_t) := -\cos \frac{\pi}{m_{st}}$$

$$\sigma_s(\mathbf{u}) = \mathbf{u} - 2\frac{B(\mathbf{u}, \mathbf{v}_s)}{B(\mathbf{v}_s, \mathbf{v}_s)}\mathbf{v}_s$$

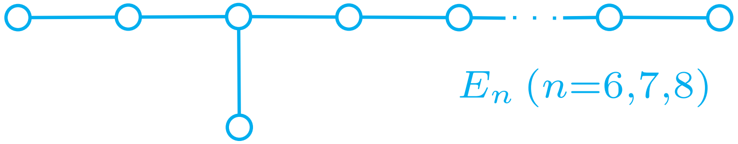
$s \mapsto \sigma_s$ gives $(W, S) \to \mathrm{GL}(V)$ reflectional representation

- Eg: W finite $\Leftrightarrow$ V Euclidean; $+$ fixes lattice in V =: Weyl group.


A_n $(n{\geq}1)$


D_n $(n{\geq}4)$


B_n $(n{\geq}2)$


E_n $(n{=}6,7,8)$

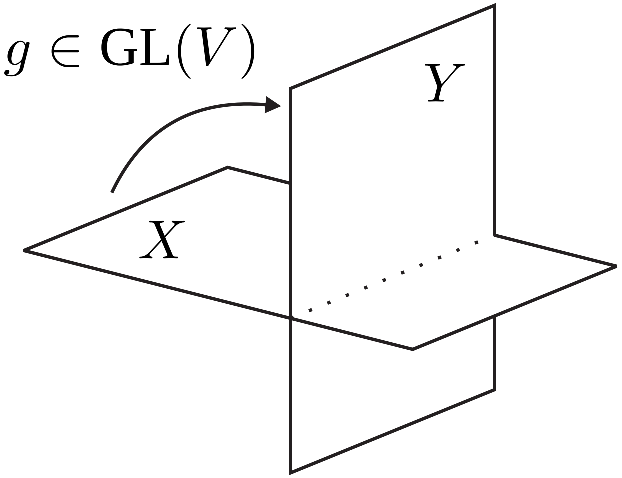

F_4


G_2

- V space over $\mathbb{F}$, $\mathrm{GL}(V)$ = group of isomorphisms $V \xrightarrow{g} V$.

- $V \supset X \xrightarrow{\alpha} Y \subset V$ partial isomorphisms between subspaces $X, Y \subset V$:

$$\alpha = g_X := \begin{cases} g \text{ on } X, \\ \text{undefined elsewhere.} \end{cases}$$



- s reflection; s_X partial reflection.

- $\mathrm{ML}(V)$:= monoid partial isomorphisms.

- reflection monoid (version 2) := $\langle$partial reflections $s_X\rangle \subset \mathrm{ML}(V)$

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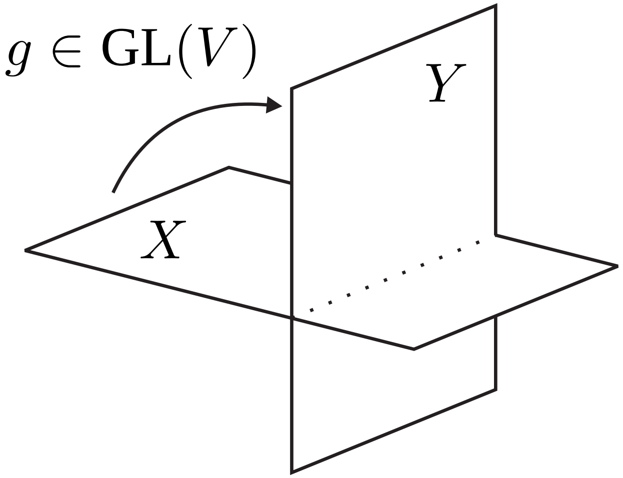

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inverse, factorizable

- reflection monoid (version 2) := $\langle$partial reflections $s_X\rangle \subset \mathrm{ML}(V)$

- $g_X f_Y = g f_{X \cap Y g^{-1}}$

$g_X = s_{1,X_1} \dots s_{k,X_k} \Rightarrow g = s_1 \dots s_k \in$ some reflection group.

- $W \subset \mathrm{GL}(V)$ reflection group

$\mathcal{B}$ system of subspaces for W:

$$:= \begin{cases} V \in \mathcal{B}, \\ X, Y \in \mathcal{B} \Rightarrow X \cap Y \in \mathcal{B}, \\ \mathcal{B}W = \mathcal{B}. \end{cases}$$

- $$\left.\begin{array}{r} \text{reflection group } W \\ \text{system } \mathcal{B} \text{ for } W \end{array}\right\} \langle g_X \,|\, g \in W, X \in \mathcal{B} \rangle \subset \mathrm{ML}(V)$$

reflection monoid (version 1) $M(W, \mathcal{B})$

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units ⟶ W

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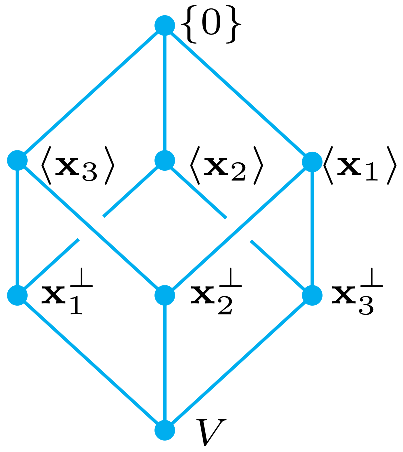
reflection monoid (version 1) $M(\underset{\underset{\text{units}}{\uparrow}}{W}, \underset{\underset{\text{idempotents}}{\uparrow}}{\mathcal{B}})$

Hyperplane arrangements

- hyperplane arrangement $\mathcal{A} \subset V$; $L(\mathcal{A})$ = intersection lattice.

- Eg: V = Euclidean space,
$\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ orthogonal basis
$\mathcal{A} = \{\mathbf{x}_i^\perp\}$ Boolean arrangement:



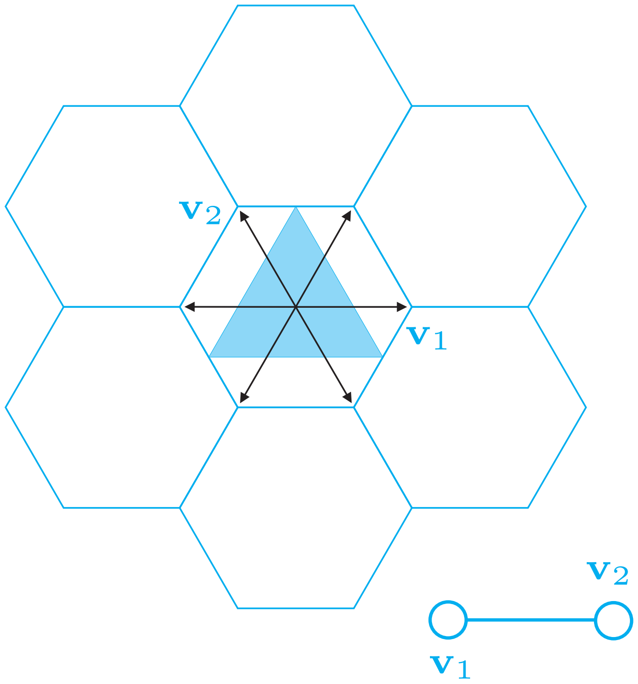
- $W \subset \mathrm{GL}(V)$ finite reflection group, $\mathcal{A}$ = reflecting hyperplanes.
reflection arrangement.

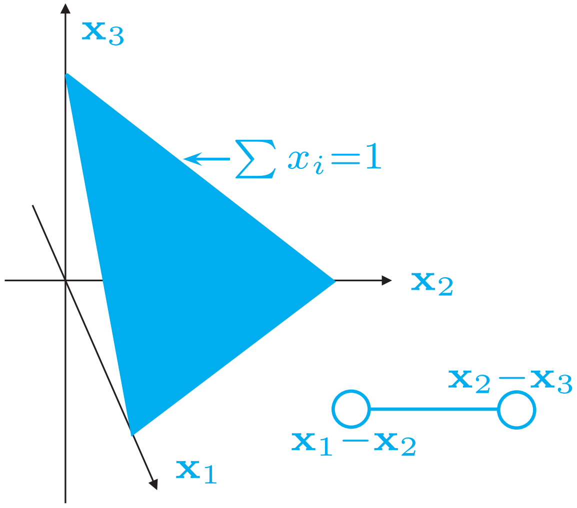
- $W \subset \mathrm{GL}(V)$ finite reflection group, $\mathcal{A} \subset V$ arrangement
$\Rightarrow \mathcal{B} = L(\mathcal{A}W)$ a system for W.

Boolean reflection monoids

$V = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_{\mathbb{R}} \hookrightarrow \{\sum x_i = 0\} \subset \widehat{V} = \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle_{\mathbb{R}}$,

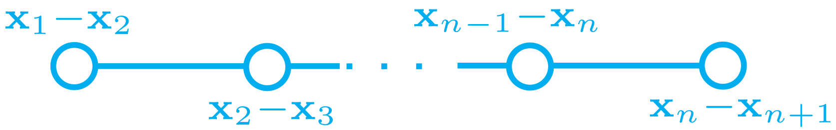
$\mathcal{A} = \{\mathbf{x}_i^{\perp}\}$ Boolean arrangement.



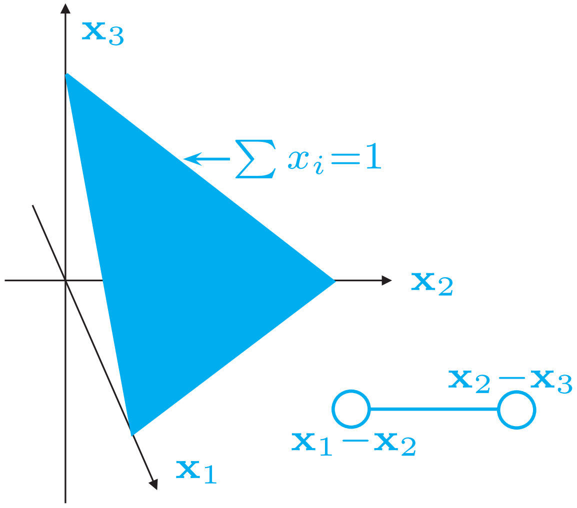


Boolean reflection monoids

$\Gamma = A_n$:



$\langle \text{nodes} \rangle_{\mathbb{R}} = V = \{\sum x_i = 0\} \hookrightarrow \widehat{V} = \langle \mathbf{x}_1, \ldots, \mathbf{x}_{n+1} \rangle_{\mathbb{R}}$



$W(\Gamma) \cong \mathfrak{S}_{n+1}$

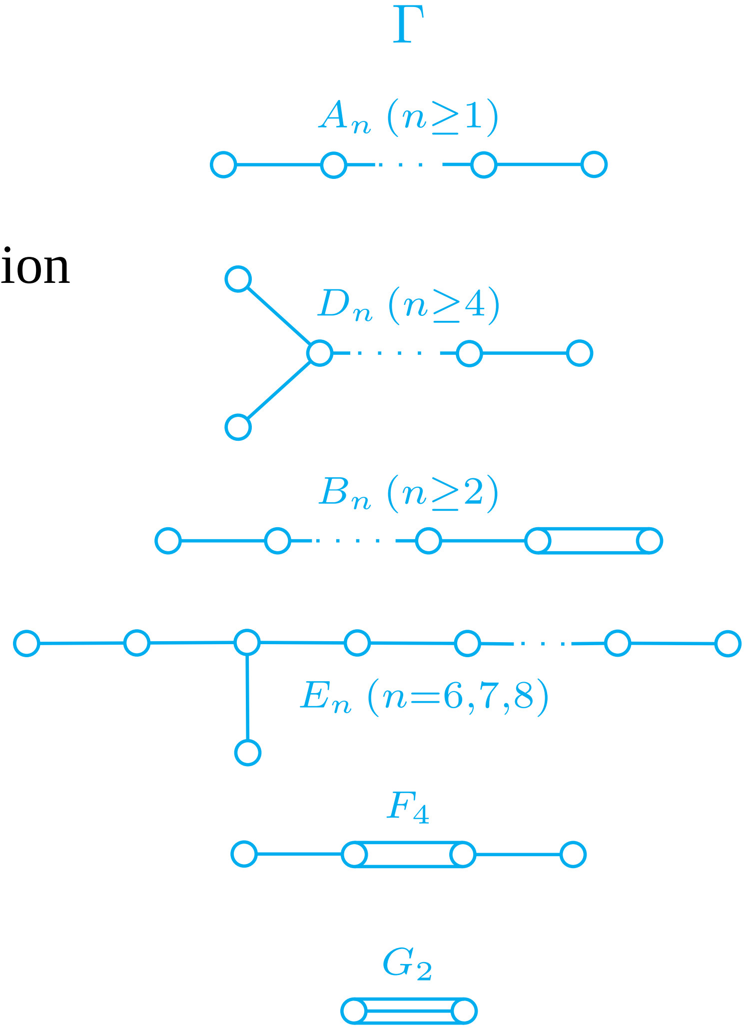
$\mathcal{A} = \{\mathbf{x}_i^{\perp}\}, \mathcal{A}W = \mathcal{A}$

$\mathcal{B} = L(\mathcal{A})$ Boolean lattice.

$M(A_n, \mathcal{B}) \cong \mathscr{I}_{n+1}$ symmetric inverse monoid.

Boolean reflection monoids

- $W = W(\Gamma) =$ Weyl group.
- $W \longrightarrow \mathrm{GL}(V)$, reflectional representation
- $V \hookrightarrow \widehat{V}$ Euclidean with orthonormal basis $\{\mathbf{x}_i\}$.
- $\mathcal{A} =$ Boolean arrangement $\{\mathbf{x}_i^{\perp}\}$, $\mathcal{B} = L(\mathcal{A}W)$.
- $M(W, \mathcal{B}) = M(\Gamma, \mathcal{B}) :=$ Boolean reflection monoid of type Γ.



Boolean reflection monoids

- $W = W(\Gamma) =$ Weyl group.

Euclidean

- $W \to \mathrm{GL}(V)$, reflectional representation
- $V \hookrightarrow \widehat{V}$ Euclidean with orthonormal basis $\{\mathbf{x}_i\}$.
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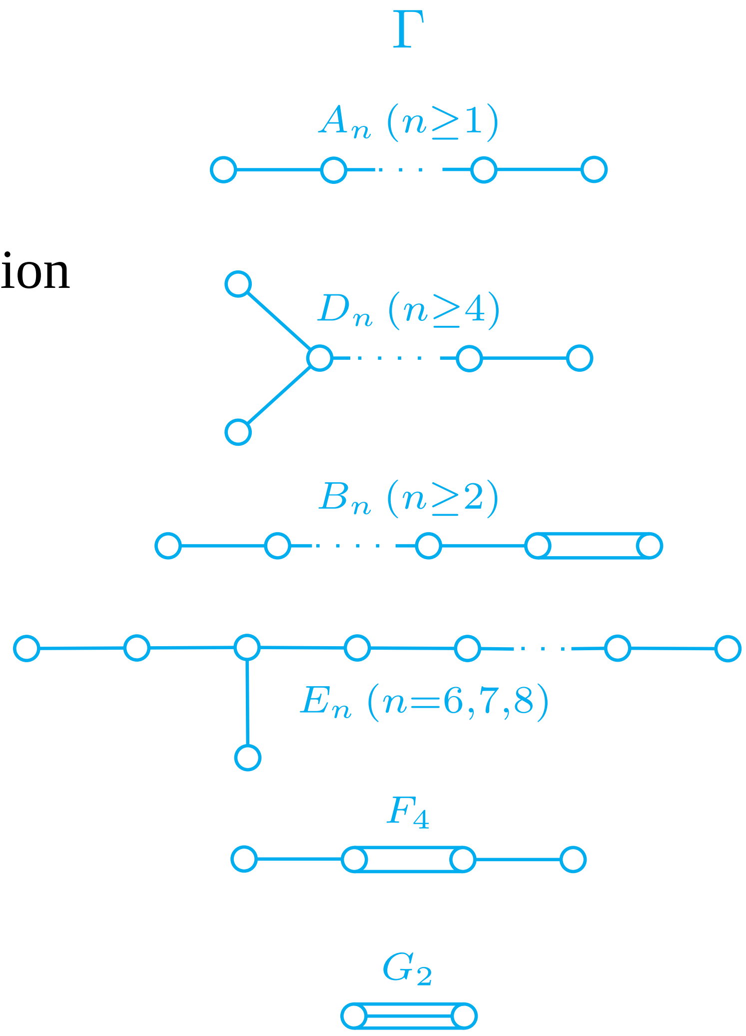
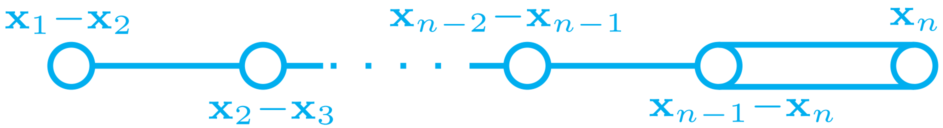


Boolean reflection monoids

$\Gamma = B_n$:



$W(B_n) \cong \mathfrak{S}_n^{\pm}$ signed permutations.

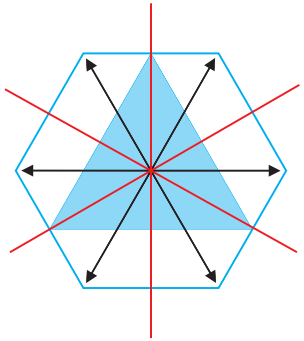
$M(B_n, \mathcal{B}) \cong \mathscr{I}_n^{\pm}$ partial signed permutations.

$\Gamma = F_4$:

$\mathcal{A} = \{\mathbf{x}_1^{\perp}, \ldots, \mathbf{x}_4^{\perp}\}$

$\Rightarrow \mathcal{A}W(F_4) = \{\mathbf{v}^{\perp} \mid \mathbf{v}$ “short root” in F_4 root system$\}$

Reflection arrangement monoids



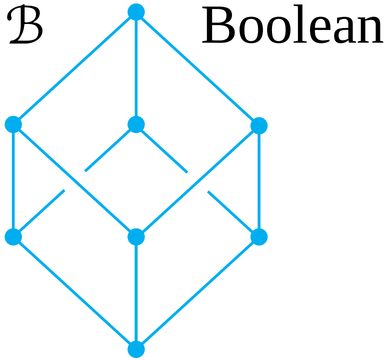
$W = W($$)$

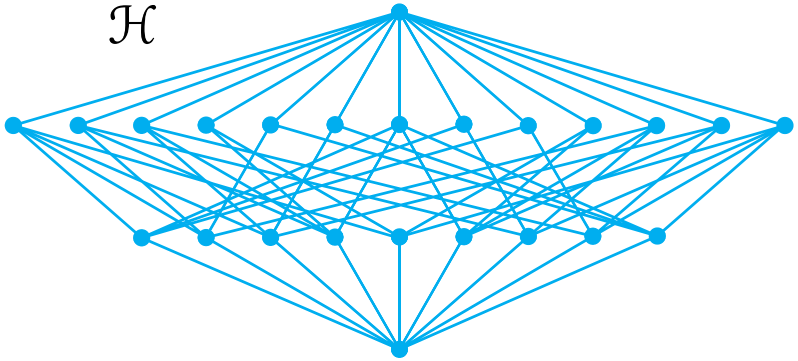
- $W \subset \mathrm{GL}(V)$ finite reflection group

  $\mathcal{A}$ = reflecting hyperplanes ($\mathcal{A}\,W = \mathcal{A}$)

  $\mathcal{H} = L(\mathcal{A})$ system for W.

- Eg: $W = W($$)$





- $M(W, \mathcal{H})$:= reflection arrangement monoid

Renner monoids

- $\mathbb{M}$ affine variety (connected)
$\mathbb{M} \times \mathbb{M} \to \mathbb{M}$ morphism of varieties

- $\mathbb{G}$ (units) $\subset \mathbb{M}$ algebraic group
  (reductive)
- $T \subset \mathbb{G}$ maximal torus

- $\mathfrak{X}(T) = \text{Hom}(T, \mathbb{F}^*)$

- Weyl group $W_{\mathbb{G}} = N_{\mathbb{G}}(T)/T$
reflection group in $\mathfrak{X}(T) \otimes \mathbb{R}$

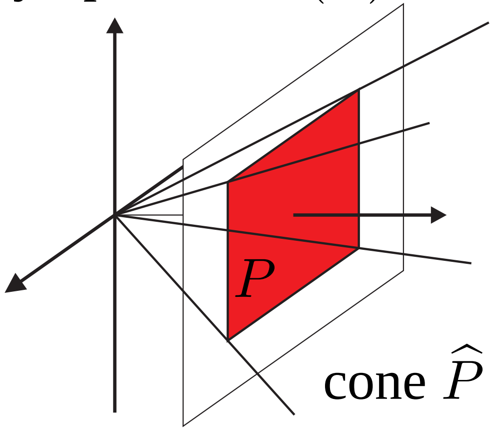
Renner monoids

• $\mathbb{M}$ affine variety (connected) $\mathbb{M} \times \mathbb{M} \to \mathbb{M}$ morphism of varieties	$\text{M}_n(\mathbb{F})$
• $\mathbb{G}$ (units) $\subset \mathbb{M}$ algebraic group (reductive)	$\text{GL}_n(\mathbb{F}) \subset \text{M}_n(\mathbb{F})$
• $T \subset \mathbb{G}$ maximal torus	$\text{D}_n^*(\mathbb{F}) =$ invertible diagonal matrices.
• $\mathfrak{X}(T) = \text{Hom}(T, \mathbb{F}^*)$	$\{\chi_1^{t_1} \dots \chi_n^{t_n} \,\vert\, t_i \in \mathbb{Z}\} \cong \mathbb{Z}^n$ $\chi_i(A) = A_{ii}$
• Weyl group $W_\mathbb{G} = N_\mathbb{G}(T)/T$ reflection group in $\mathfrak{X}(T) \otimes \mathbb{R}$	$W_\mathbb{G} =$ permutation matrices $\cong \mathfrak{S}_n$

Renner monoids $R_{\mathbb{M}} := \overline{N_{\mathbb{G}}(T)}/T = \{x \in \mathbb{M} \,|\, xT = Tx\}.$

- $T \subset \overline{T}, E(\overline{T}) =$ idempotents

- polytope $P \subset \mathfrak{X}(T) \otimes \mathbb{Q}$



$\mathfrak{X}(\overline{T}) = \mathfrak{X}(T) \cap \widehat{P}$
$E(\overline{T}) \cong$ face lattice of $\widehat{P}$

$$\mathcal{A} = \{\langle F \rangle_{\mathbb{R}} \,|\, F \in \widehat{P} \text{ top dimensional}\}$$
$\mathcal{B} = L(\mathcal{A})$ system for $W_{\mathbb{G}}$; consider $M(W_{\mathbb{G}}, \mathcal{B})$

Renner monoids

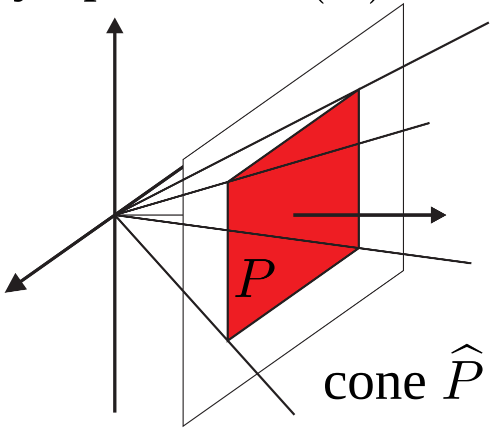
$$R_{\mathbb{M}} := \overline{N_{\mathbb{G}}(T)}/T = \{x \in \mathbb{M} \,|\, xT = Tx\}.$$

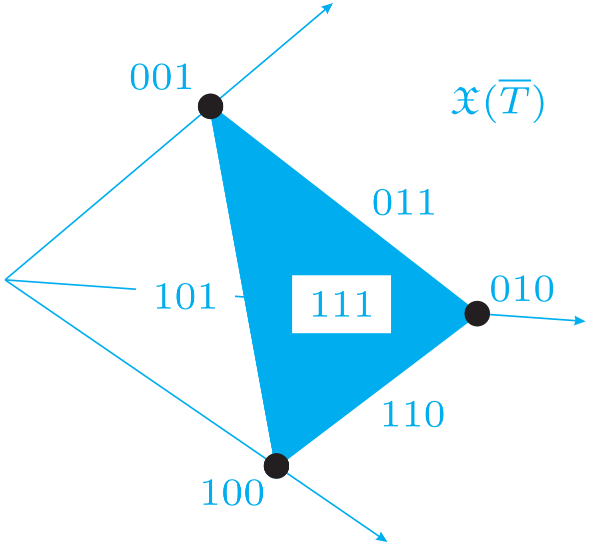
- $T \subset \overline{T}, E(\overline{T})$ = idempotents

$\mathrm{D}_n(\mathbb{F})$ = diagonal matrices

$E(\overline{T}) = \{A \,|\, A_{ii} = 0, 1\}$

- polytope $P \subset \mathfrak{X}(T) \otimes \mathbb{Q}$





$\mathfrak{X}(\overline{T}) = \mathfrak{X}(T) \cap \widehat{P}$

$E(\overline{T}) \cong$ face lattice of $\widehat{P}$

$\mathcal{A} = \{\langle F \rangle_{\mathbb{R}} \,|\, F \in \widehat{P}$ top dimensional$\}$

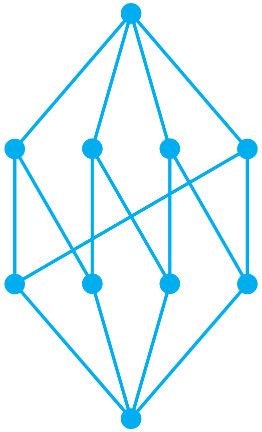
$\mathcal{B} = L(\mathcal{A})$ system for $W_{\mathbb{G}}$; consider $M(W_{\mathbb{G}}, \mathcal{B})$

Renner monoids

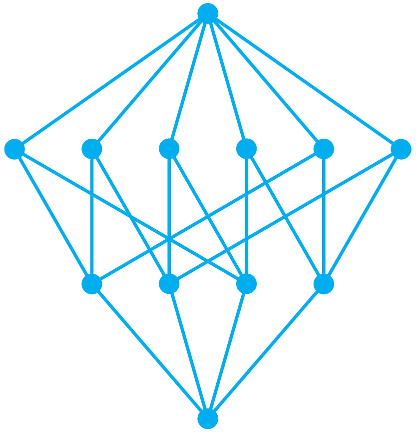
- **Theorem:** there is a surjective homomorphism $M(W_{\mathbb{G}}, \mathcal{B}) \to R_{\mathbb{M}}$. An isomorphism $\Leftrightarrow \widehat{P}$ a simplicial cone.

- **Eg:** $\mathbb{M} = \overline{\text{Ad}(\mathbb{G})\mathbb{F}^*}$, $\mathbb{G} =$ adjoint simple group type B_2, $\text{Ad} : \mathbb{G} \to \text{GL}(\mathfrak{g})$ adjoint representation.

$E(R_{\mathbb{M}})$



$\mathcal{B}$



Renner monoids

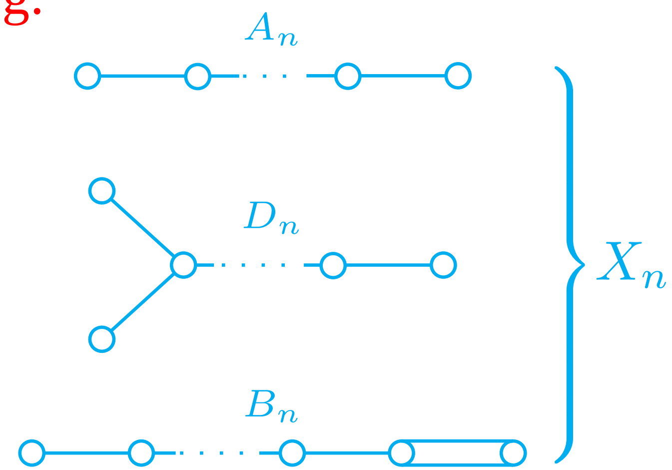
- **Theorem:** $R_{\mathbb{M}}$ Renner monoid, $\rho : R_{\mathbb{M}} \longrightarrow \mathrm{ML}(V)$ injective, with $\rho R_{\mathbb{M}}$ a reflection monoid and $W_{\mathbb{G}} \subset R_{\mathbb{M}}$ acting essentially on V $\Rightarrow W_{\mathbb{G}} = A_n (n > 1)$, D_n (n odd) or E_6.

Orders

- $W(\Gamma)$ a Weyl group $\Rightarrow \sum_{\Psi\subset\Gamma}(-1)^{|\Psi|}[W(\Gamma):W(\Psi)]=1.$
- Theorem. $W\subset \mathrm{GL}(V)$, $\mathcal{B}$ system for W,

$$|M(W,\mathcal{B})|=\sum_{X\in\mathcal{B}}[W:W_X]$$

- Eg:



$$W = \text{Weyl group type } X_n$$

$$\mathcal{B}=L(\mathcal{A}),\ \mathcal{A}=\text{Boolean}$$

$$|M(W,\mathcal{B})|=$$

$$\sum_k \binom{n}{k}[W(X_n):W(X_k)]$$

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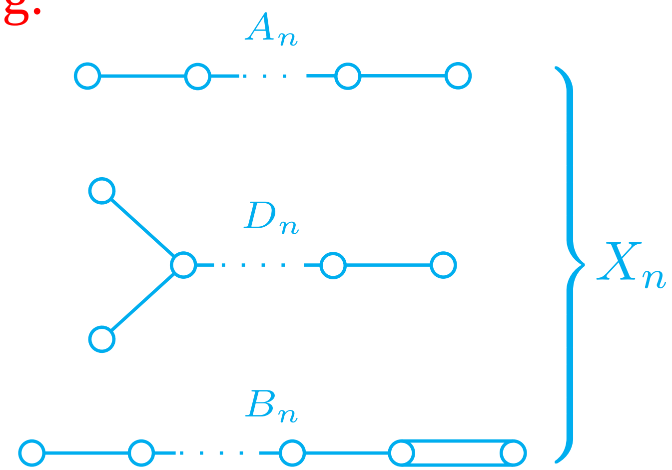

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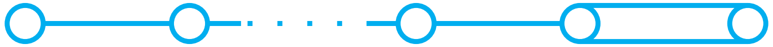
Orders

- **Eg:** $\Gamma = B_n$: 

$\mathcal{H}$ = intersection lattice of reflecting hyperplanes

- $q \in \mathbb{Z}^{>0}$, $\lambda = (\lambda_1, \ldots, \lambda_p)$ partition of q.

$\Leftrightarrow 1 \leq \lambda_1 \leq \cdots \leq \lambda_p$ with $\sum \lambda_i = q$.

$b_\lambda := b_1!b_2! \ldots (1!)^{b_1}(2!)^{b_2} \ldots$ $\qquad d_\lambda = 4^p b_\lambda \lambda_1! \ldots \lambda_p!$

b_i = number of λ_j that equal i.

- **Theorem.** $|M(B_n, \mathcal{H})| = 2^{2n-1}(n!)^2 \sum_{m,\lambda} \frac{1}{4^m\, d_\lambda}$

the sum over all $0 \leq m \leq n$ and partitions λ of $n - m$.

Presentations

- $W = \langle S \rangle$, $\mathcal{B} = L(\mathcal{A}W)$.

- **Theorem.** $M(W, \mathcal{B})$ has presentation,

  generators: $s \in S$ and $\varepsilon_X (X \in \mathcal{A}/W)$

  (fix $\widehat{\varepsilon}_Y := g^{-1}\varepsilon_X g$ for $Y \in \mathcal{A}W$; fix $Z = \bigcap Y_i$ $(*)$ for $Z \in \mathcal{B}$, $Y_i \in \mathcal{A}W$, $\widehat{\varepsilon}_Z := \prod \widehat{\varepsilon}_{Y_i}$)

  relations: relations for W,

  ε_X's commuting idempotents,

  $\widehat{\varepsilon}_Z = \prod \widehat{\varepsilon}_{Y_i}$, $Z \in \mathcal{B}/W$, $Z = \bigcap Y_i$ "different" from $(*)$,

  $s\widehat{\varepsilon}_Y = \widehat{\varepsilon}_{(Y)s} s$ for $(s, Y) \in S \times \mathcal{A}W$,

  a "small" number of $\widehat{\varepsilon}_Z g = \widehat{\varepsilon}_Z$ for $g \in W_Y$.

Presentations

- Eg: $\Gamma = A_n$ 

$W = W(\Gamma)$ Weyl group, $\mathcal{B}$ = Boolean system.

$M(W, \mathcal{B}) \cong \mathscr{I}_{n+1}$ symmetric inverse monoid.

$$\begin{aligned} M(A_n, \mathcal{B}) = \langle s_1, \ldots, s_n, \varepsilon \mid\ & (s_i s_j)^{m_{ij}} = 1, \\ & \varepsilon^2 = \varepsilon, \\ & \varepsilon s_n \varepsilon s_n = s_n \varepsilon s_n \varepsilon, \\ & s_i \varepsilon = \varepsilon s_i\ (i \neq n), \\ & s_n \varepsilon s_n \varepsilon s_n = s_n \varepsilon s_n \varepsilon \rangle \end{aligned}$$

[Popova 1961]