Knot invariants: natural and not

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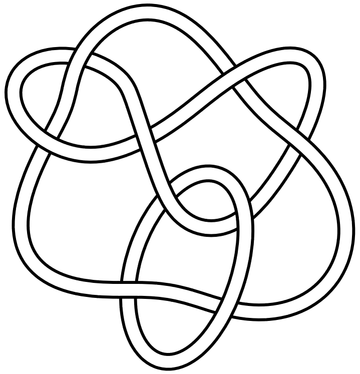
(Hopefully) the contents

- Natural invariants.
- Not-natural invariants.
- Making Jones natural.
- Coloured posets.

- Link: $L := \coprod S^1 \subset S^3$ (as submanifold).
  $L_1 \approx L_2 \Leftrightarrow S^3 \xrightarrow{f} S^3$ orientation preserving with $f(L_1) = f(L_2)$.

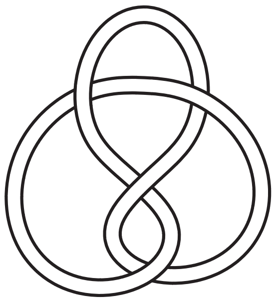
- diagrams: $L_1 \approx L_2 \Leftrightarrow$ any two diagrams for the L_i related by Reidemeister moves

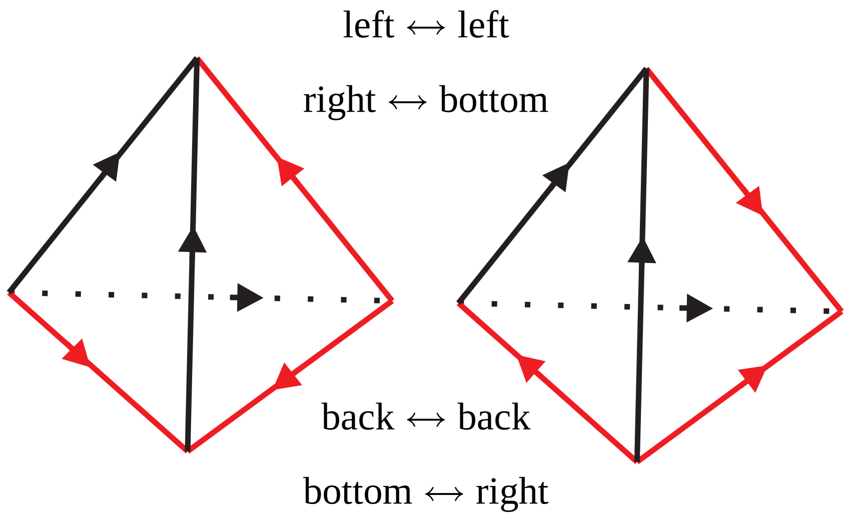


- invariants: well defined labelling of equivalence classes!

- label by homeomorphism type of $S^3 - L$.

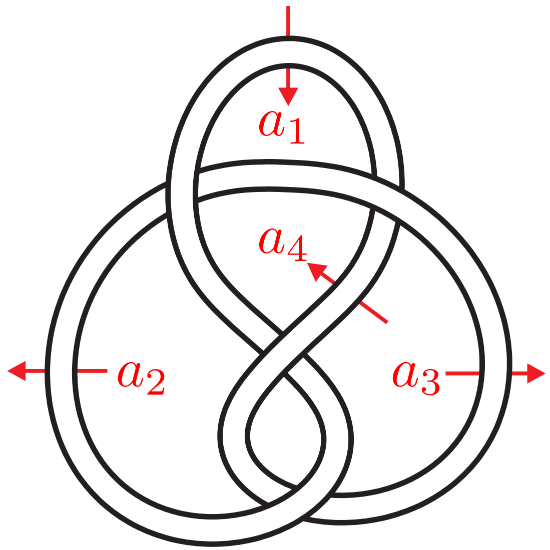
- Eg:

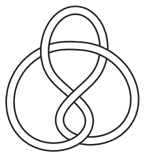
$S^3 -$  $\approx$



- **Theorem** [Gordon-Luecke 1989]: the homeomorphism type of the complement of a knot is a complete invariant.

- label by fundamental group $\pi_1(S^3 - L)$ (=: knot group)
- **Theorem** [Whitten, Gonzales-Acuna]: complete invariant for prime knots.
- **Theorem** [Mostow, Gromov]: complete invariant for hyperbolic knots.
- Wirtinger presentation:



$\pi_1\Big(S^3 -$  $\Big) = \langle a_1, a_2, a_3, a_4 \mid a_1a_3 = a_2a_1,$

$$a_3a_1 = a_4a_3,$$
$$a_2a_4 = a_4a_1,$$
$$a_2a_3 = a_4a_2\rangle$$

Natural invariants: π_1 (the bad news)

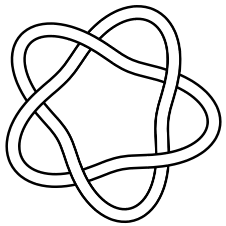
- A group is large if it has a finite index subgroup that surjects a non-abelian free group.

- **Theorem** [Cooper, Long, Reid 1997]: "most" prime knots have large knot groups.

Natural invariants: homology

	H_0	H_1	H_2	H_3

Natural invariants: homology

	H_0	H_1	H_2	H_3
				

Natural invariants: homology

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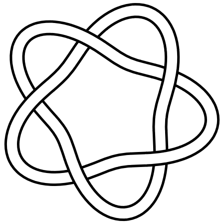	$\mathbb{Z}$	$\mathbb{Z}$	0	0

Natural invariants: homology

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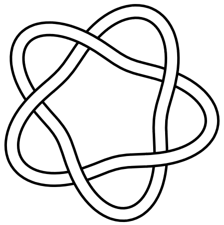
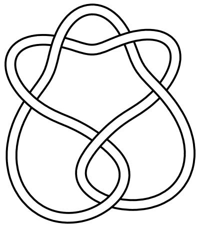
	$\mathbb{Z}$	$\mathbb{Z}$	0	0
				

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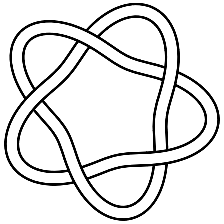
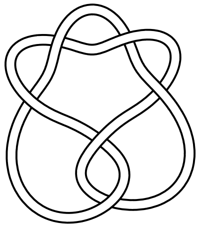
	$\mathbb{Z}$	$\mathbb{Z}$	0	0
	$\mathbb{Z}$	$\mathbb{Z}$	0	0

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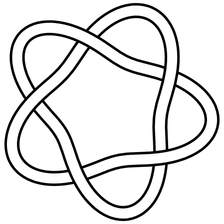
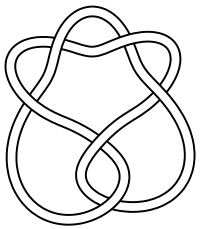
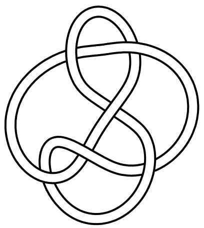
	$\mathbb{Z}$	$\mathbb{Z}$	0	0
	$\mathbb{Z}$	$\mathbb{Z}$	0	0
				

Natural invariants: homology

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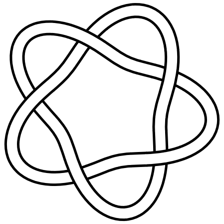
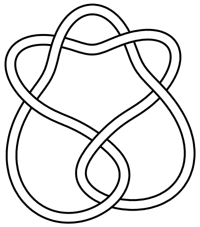
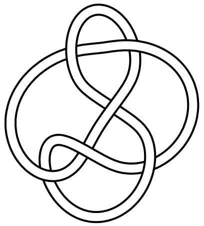
	$\mathbb{Z}$	$\mathbb{Z}$	0	0
	$\mathbb{Z}$	$\mathbb{Z}$	0	0
	$\mathbb{Z}$	$\mathbb{Z}$	0	0

Natural invariants: homology

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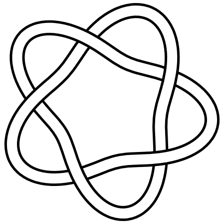
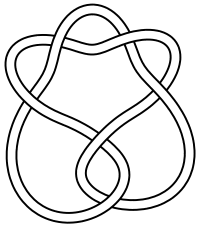
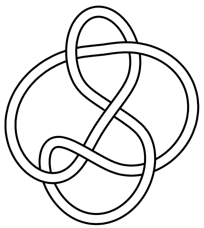
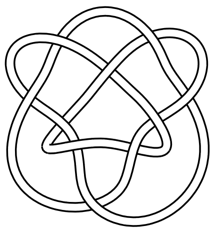
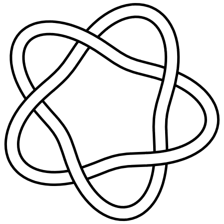
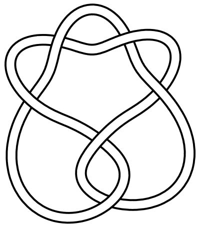
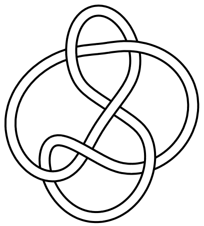
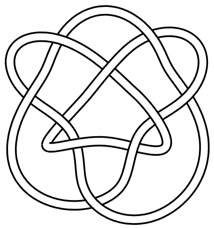
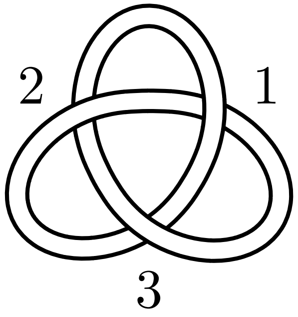
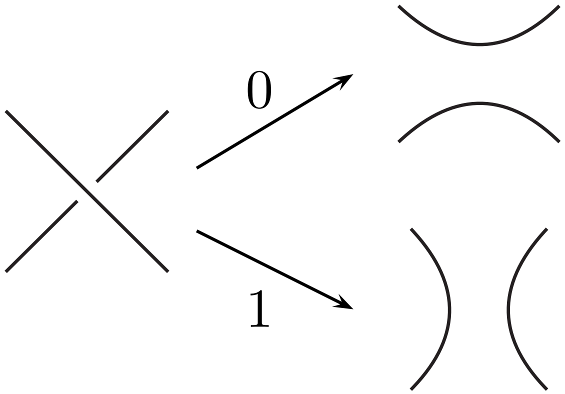
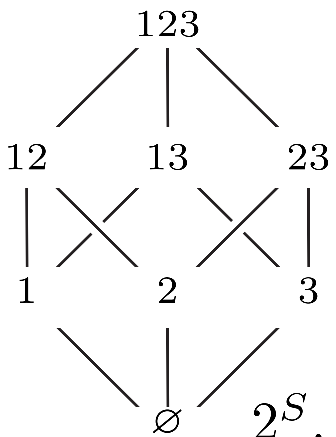
	$\mathbb{Z}$	$\mathbb{Z}$	0	0
	$\mathbb{Z}$	$\mathbb{Z}$	0	0
	$\mathbb{Z}$	$\mathbb{Z}$	0	0
				

Natural invariants: homology

	H_0	H_1	H_2	H_3
	$\mathbb{Z}$	$\mathbb{Z}$	0	0
	$\mathbb{Z}$	$\mathbb{Z}$	0	0
	$\mathbb{Z}$	$\mathbb{Z}$	0	0
	$\mathbb{Z}$	$\mathbb{Z}$	0	0





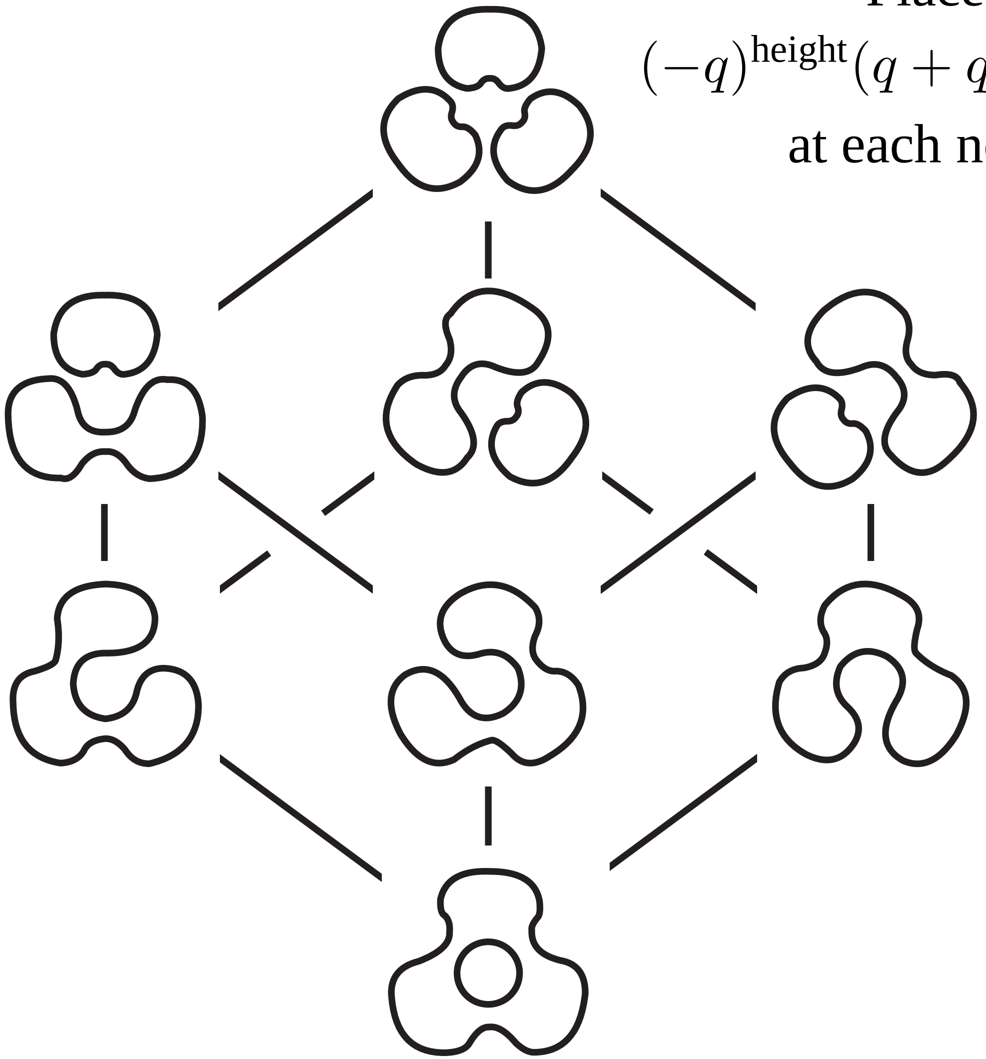


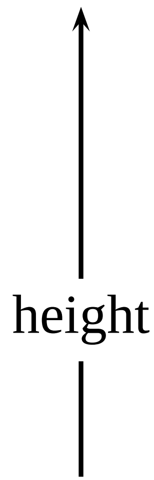
$2^S, S = \{1, 2, 3\}$

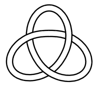
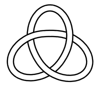
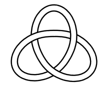
Place

$(-q)^{\text{height}}(q + q^{-1})^{\#\text{circles}}$

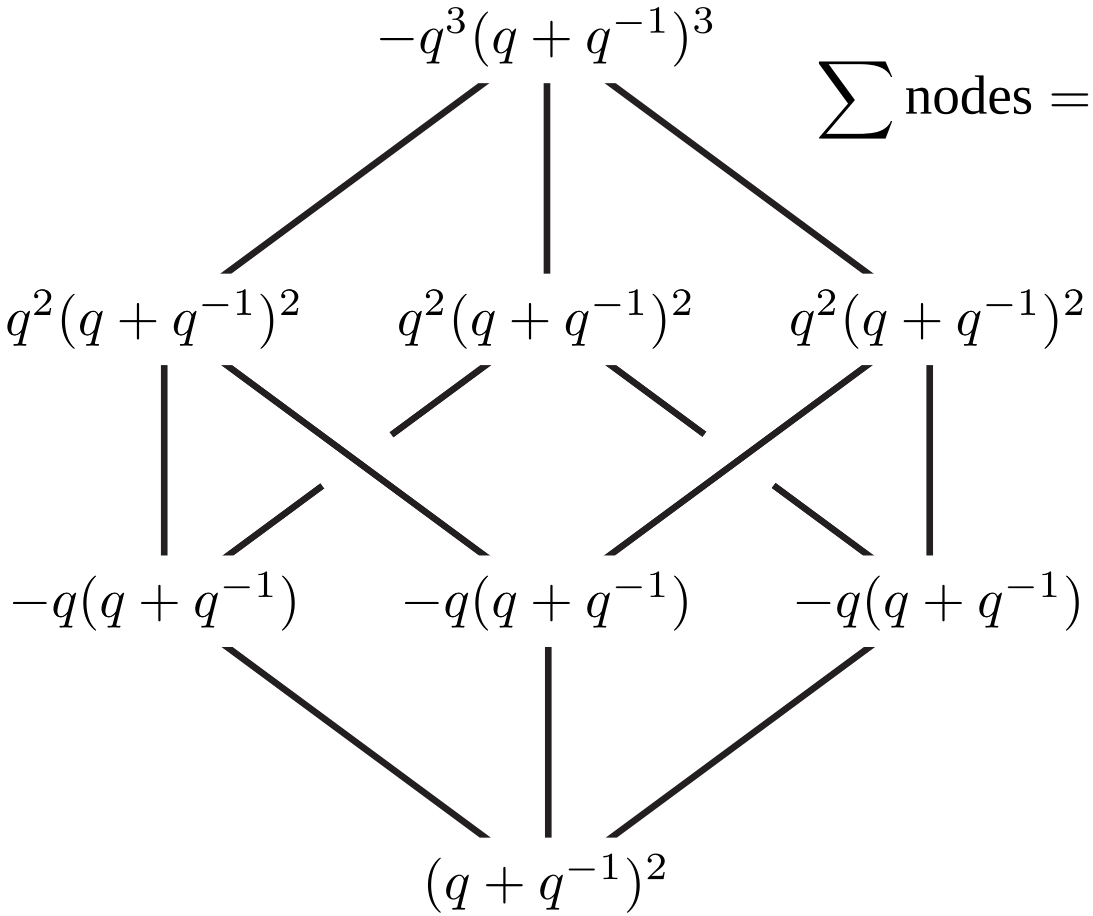
at each node

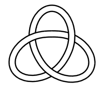


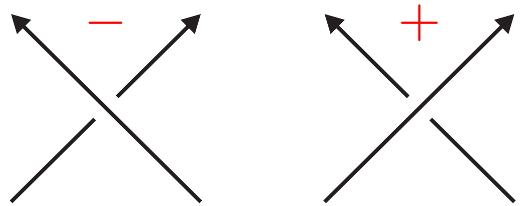


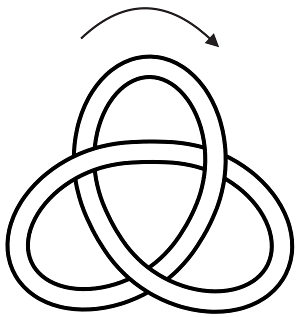
$J\Big($  $\Big) = \dfrac{1}{(q+q^{-1})}\widehat{J}\Big($  $\Big) \longleftarrow (-1)^{n_-} q^{n_+ - 2n_-}\Big\langle$  $\Big\rangle$

(Jones) (unnormalized Jones) (Kauffman bracket)



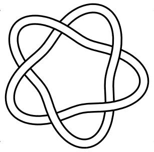
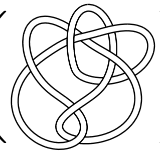
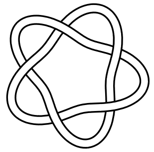
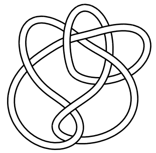
$\sum$ nodes $= \Big\langle$  $\Big\rangle = -q^6 + q^2 + 1 + q^{-2}$





Not natural invariants: the Jones polynomial

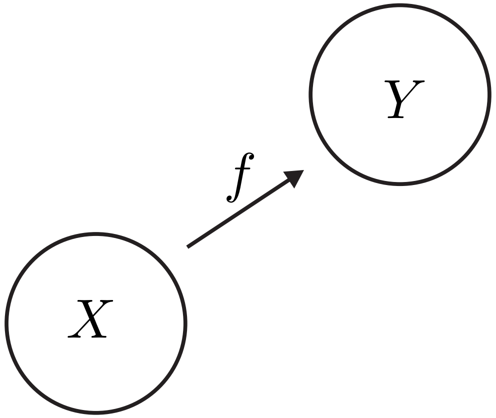
- Easy to calculate!

- Not complete: $J\Big($  $\Big) = J\Big($  $\Big)$ but  $\neq$ 

- Conjecture: $J(K) = J(\text{unkot}) \Rightarrow K = \text{unknot}$ (seems pretty unlikely)

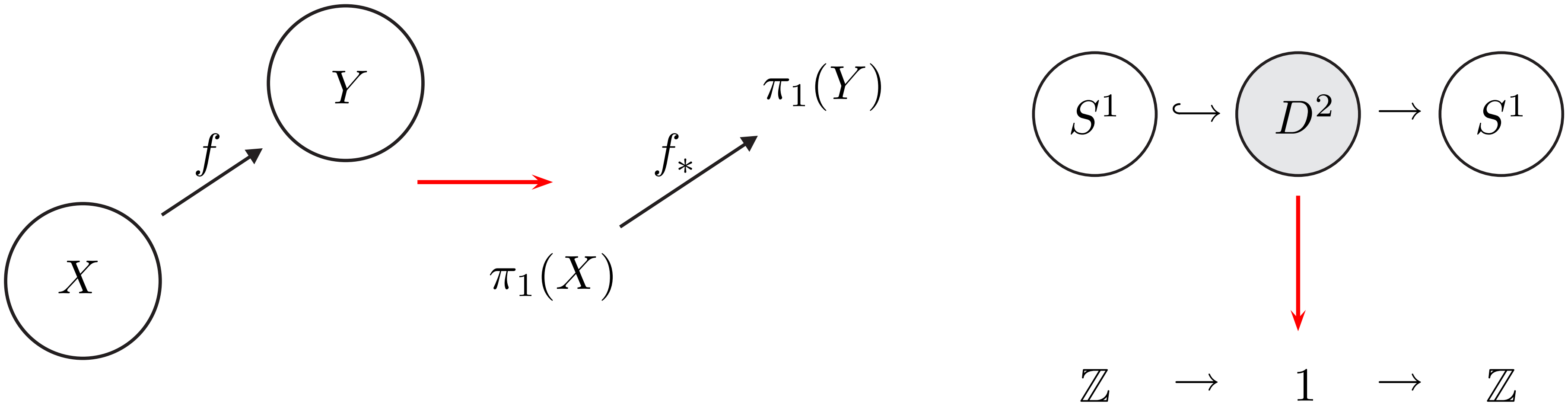
Natural versus not natural: categorification

- Eg: naturality of π_1:



- Khovanov's categorification of the Jones polynomial

- Eg: naturality of π_1:



- Khovanov's categorification of the Jones polynomial

Formal nonsense: graded spaces

$$\begin{array}{ccccccc} \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \end{array}$$

- $V = \cdots \oplus V_{-2} \oplus V_{-1} \oplus V_0 \oplus V_1 \oplus V_2 \oplus \cdots$ (V_i = vector spaces over k)

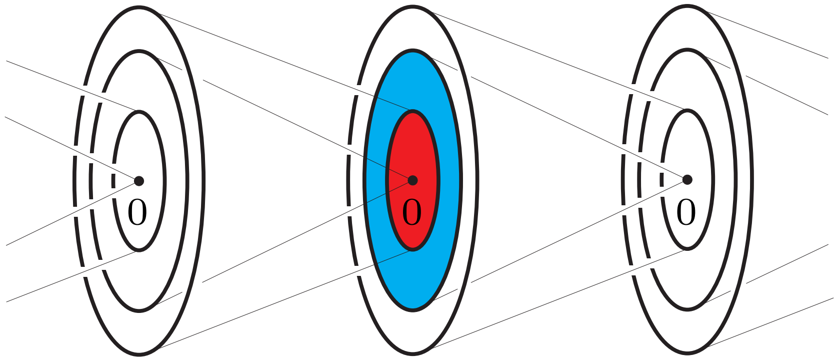
$$\begin{array}{ccccccc} \cdots & -2 & -1 & 0 & 1 & 2 & \cdots \end{array}$$

- $V[k] = \cdots \oplus V_{-k-2} \oplus V_{-k-1} \oplus V_{-k} \oplus V_{-k+1} \oplus V_{-k+2} \oplus \cdots$

(degree shift)

- tensor product $V \otimes U = \bigoplus (V \otimes U)_k$ with $(V \otimes U)_k = \bigoplus_{i+j=k} V_i \otimes U_j$
- graded dimension $q\dim V := \sum \dim V_j q^j \in \mathbb{Z}[q, q^{-1}]$
- $q\dim(U \otimes V) = q\dim U \times q\dim V$

  $q\dim V[k] = q^k \times q\dim V$

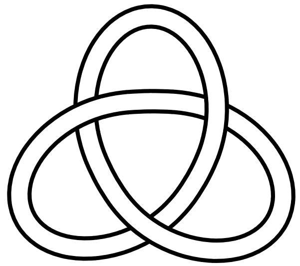


U_i graded spaces

image $\partial_{i+1} \subseteq$ kernel ∂_i

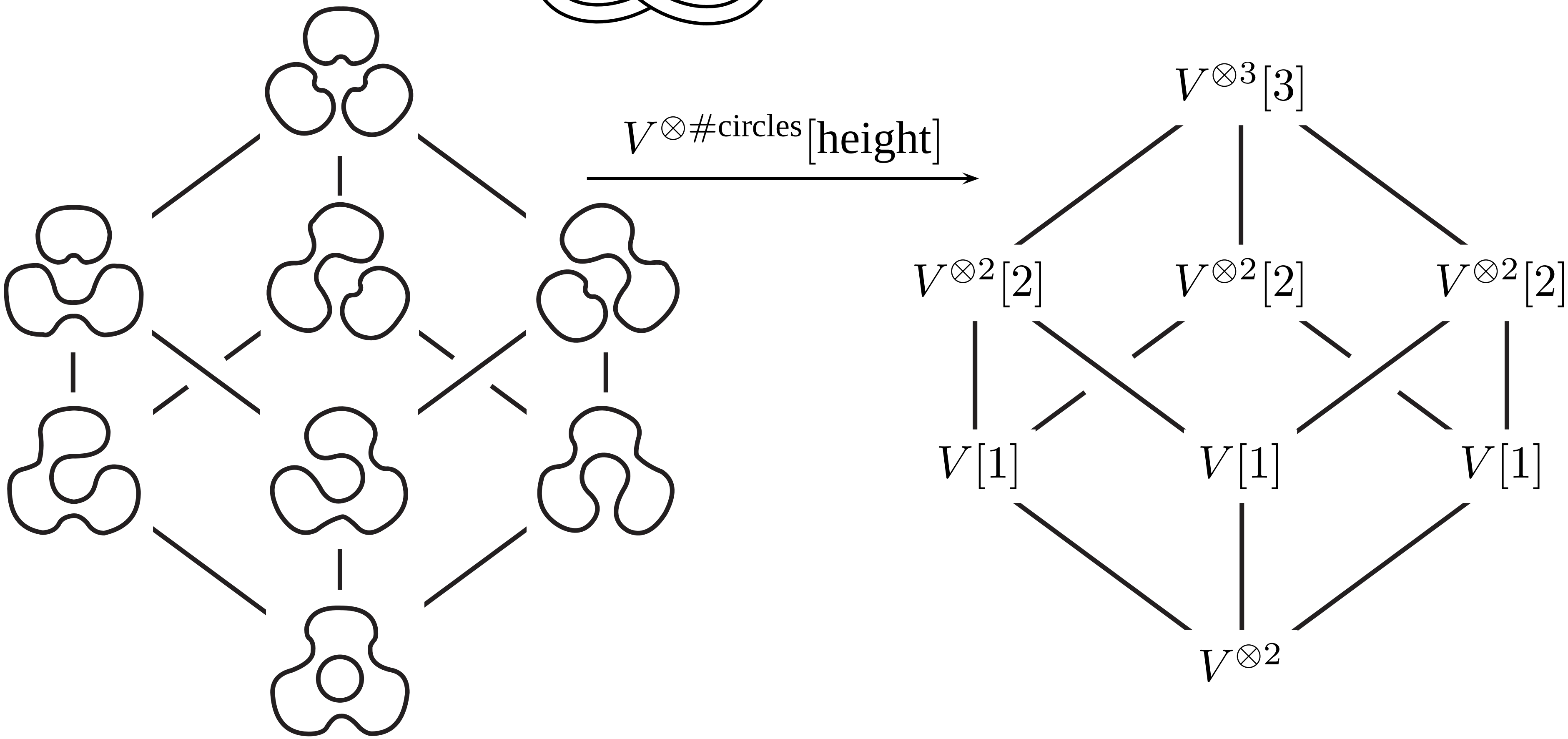
$$\mathcal{C}_* \qquad \longrightarrow U_{i+1} \xrightarrow{\partial_{i+1}} U_i \xrightarrow{\partial_i} U_{i-1} \longrightarrow \qquad \partial^2 = 0$$

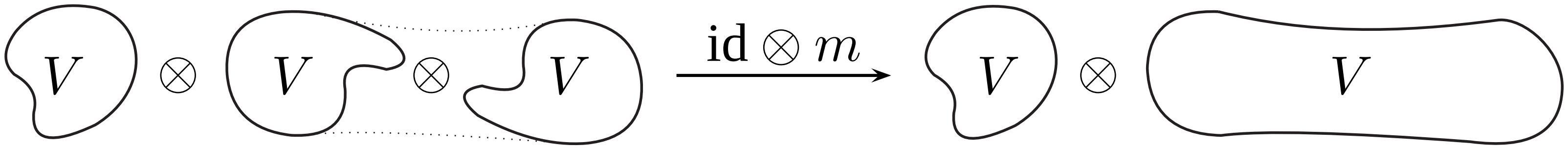
- homology $H_i(\mathcal{C}_*) = \dfrac{\text{kernel } \partial_i}{\text{image } \partial_{i+1}}$ (inherits grading from U_i)
- graded Euler characteristic $\chi(\mathcal{C}_*) = \sum (-1)^i q\dim H_i(\mathcal{C}_*)$



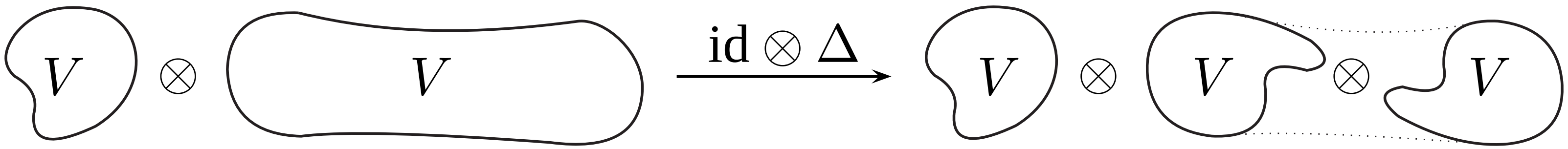
$$V = \mathbb{Q} \oplus \mathbb{Q}$$

$$\quad -1 \qquad 1$$

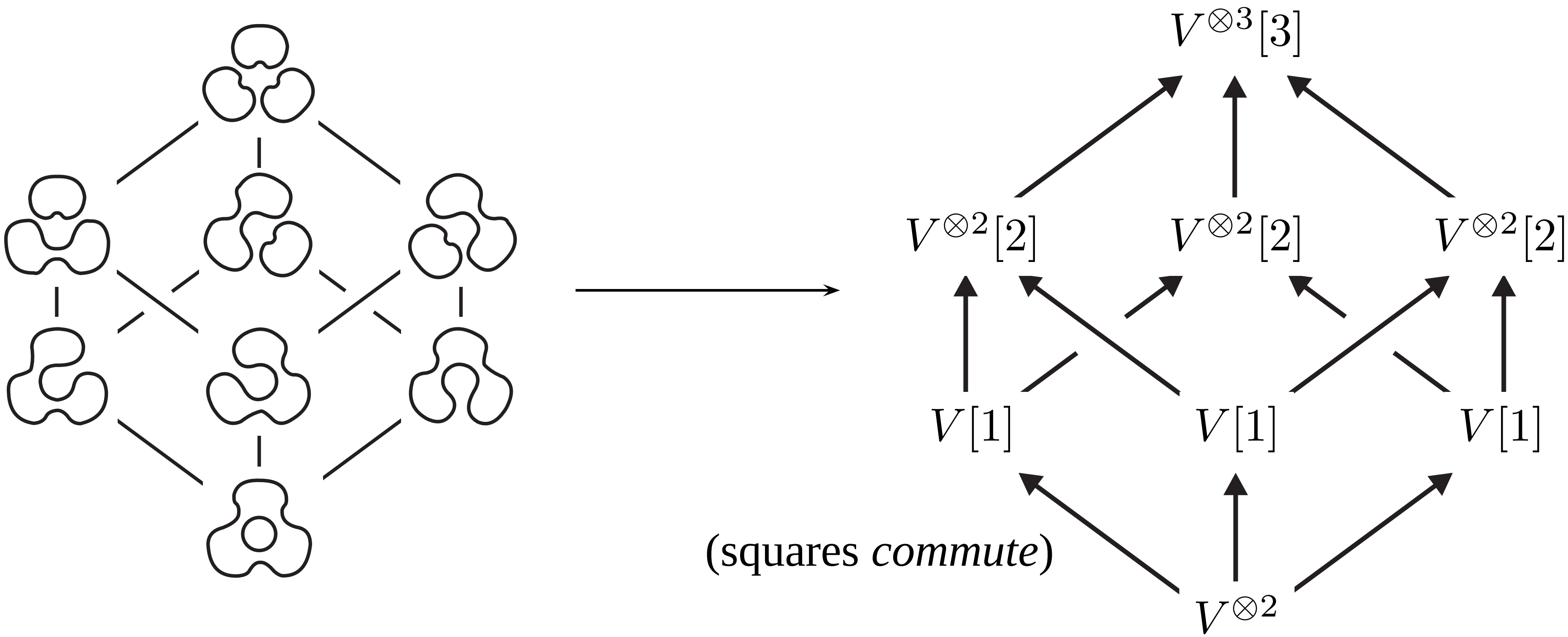




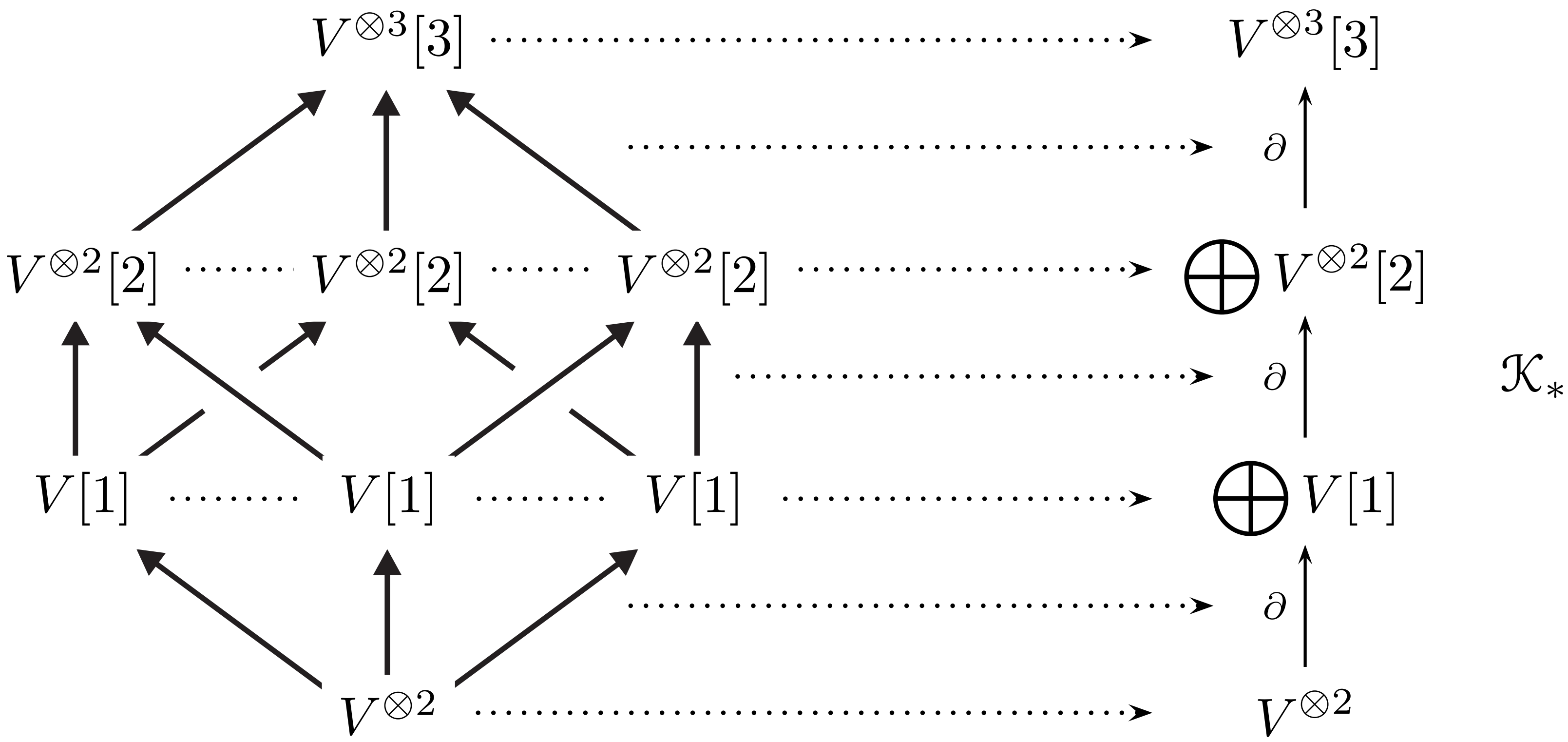
V ⊗ V ⊗ V
$\mathrm{id} \otimes m$
V ⊗ V



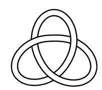
V ⊗ V
$\mathrm{id} \otimes \Delta$
V ⊗ V ⊗ V

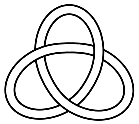


$V^{\otimes 3}[3]$
$V^{\otimes 2}[2]$
$V^{\otimes 2}[2]$
$V^{\otimes 2}[2]$
$V[1]$
$V[1]$
$V[1]$
$V^{\otimes 2}$
(squares *commute*)

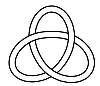


add $\pm$'s to edge maps so squares *anticommute*

Khovanov homology $KH_*\Big($  $, \mathbb{Q}\Big) = H_*(\mathcal{K}_*)$

	6	4	2	0	-2	$q\dim$
KH_0	$\mathbb{Q}$					q^6
KH_1			$\mathbb{Q}$			q^2
KH_2						0
KH_3				$\mathbb{Q}$	$\mathbb{Q}$	$1+q^{-2}$

Euler characteristic $\chi(\mathcal{K}_*)$

$$= \sum(-1)^i q\dim KH_i \left(\right.$$  $$\left. , \mathbb{Q} \right)$$

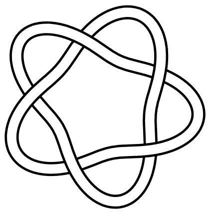
$$= q^6 - q^2 - 1 - q^{-2}$$

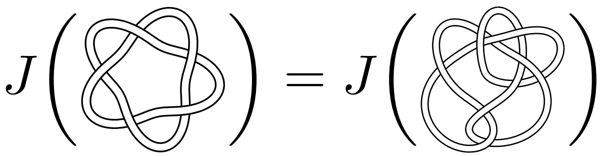
key property 1, essentially by construction:

$$(-1)^{1+n_-} q^{n_+ - 2n_-} \chi(\mathcal{K}_*) = \widehat{J}(K)$$

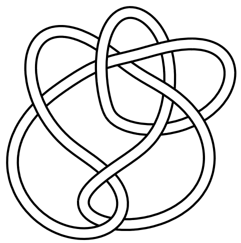
key property 2, and minor miracle: KH_* an invariant (after a bit of nudging)

$\mathbb{Q}$						
		$\mathbb{Q}$				
		$\mathbb{Q}$				
				$\mathbb{Q}$		
					$\mathbb{Q}$	$\mathbb{Q}$





$\mathbb{Q}$							
		$\mathbb{Q}$					
		$\mathbb{Q}$					
			$\mathbb{Q}$	$\mathbb{Q}$			
			$\mathbb{Q}$		$\mathbb{Q}$		
					$\mathbb{Q}\oplus\mathbb{Q}$		
							$\mathbb{Q}$
						$\mathbb{Q}$	$\mathbb{Q}$



Other categorifications (or: just like the Jones)

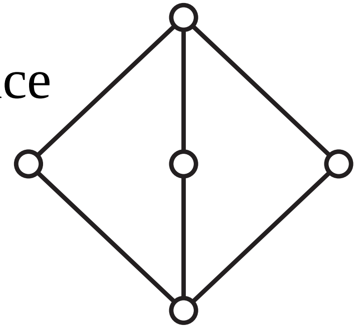
- Alexander polynomial: Heegaard-Floer homology (Ozsváth and Szabó)
- HOMFLY polynomial: Khovanov-Rozansky homology
- chromatic polynomial: graph homology (Helme-Guizon and Rong)

reflexive,anti-symmetric, transitive

- Poset $(P, \leq)$

set

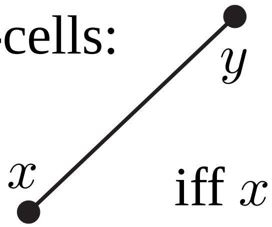
Eg: Braid lattice



- order (simplicial) complex $\Delta(P)$:

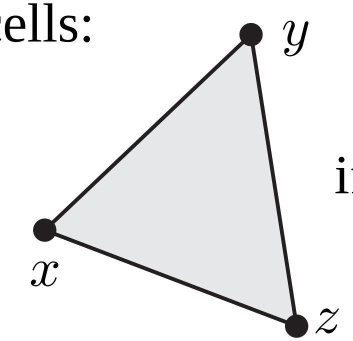
0-cells: P

1-cells:



iff $x < y$

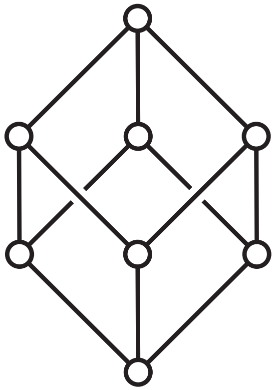
2-cells:

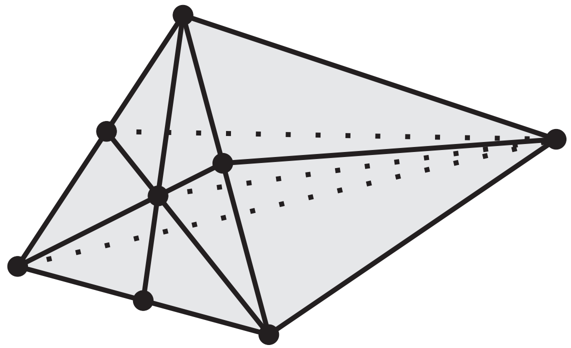


iff $x < y < z$

n-cells: $\sigma = x_0 < \cdots < x_n$

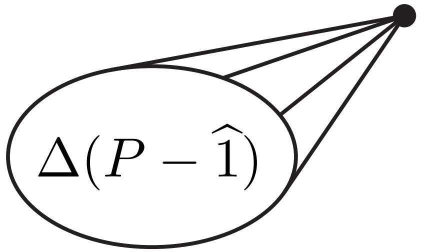
- Eg:

$P =$ 

$\Delta(P) =$ 

Topology of posets 2

- P has a $\widehat{1}$
$\Rightarrow \Delta(P) \approx$ cone on $\Delta(P - \widehat{1})$.



- simplicial chain complex of Δ: $\cdots \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots$

$$C_n := \mathbb{Z}^{\#n\text{-cells}} = \{\sum \lambda_\sigma \sigma\}_{\lambda_\sigma \in \mathbb{Z}}$$
$$\partial(\lambda_\sigma \, x_0 < \cdots < x_n) := \sum (-1)^j \lambda_\sigma (x_0 < \cdots < \widehat{x}_j < \cdots < x_n)$$

- order homology of poset P: $H_*(P, \mathbb{Z}) := H_*(\Delta(P - \widehat{0}, \widehat{1}), \mathbb{Z})$

↑ Folkman complex

- **Theorem** [Folkman-Björner]: P finite geometric lattice of rank r

$$H_n(P,\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0, \\ \mathbb{Z}^{|\mu(\widehat{0},\widehat{1})|} & n=r-2, \\ 0 & \text{otherwise.} \end{cases}$$

- Möbius function μ of a poset P: $\mu := \zeta^{-1}$ in incidence algebra of P

- Eg: $P = (\mathbb{Z}^{>0}, \leq)$ with $m \leq n$ iff $m|n$
  $\mu =$ classical number-theoretic Möbius function.

- $$\left.\begin{array}{c}\text{simplex } \sigma\in\Delta\\ \text{face } \tau\subset\sigma\end{array}\right\}\xrightarrow{\mathscr{A}}\left\{\begin{array}{c}A_\sigma\in R\text{-Mod}\\ f_\sigma^\tau: A_\sigma\to A_\tau\end{array}\right.$$

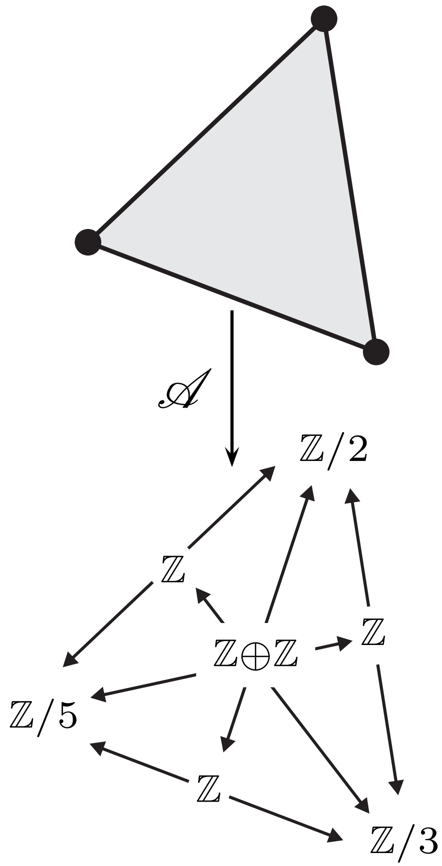
$(\Delta,\mathscr{A})$ system of local coefficients

- simplicial chain complex with local coefficients:

$$\cdots\xrightarrow{\partial}C_{n+1}\xrightarrow{\partial}C_n\xrightarrow{\partial}C_{n-1}\xrightarrow{\partial}\cdots$$

$$C_n=\{\sum\lambda_\sigma\sigma\}_{\lambda_\sigma\in A_\sigma}$$

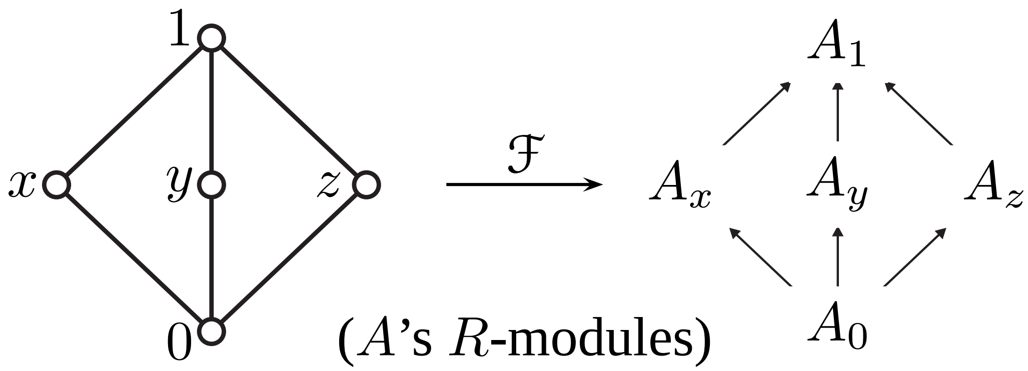
$$\partial(\lambda_\sigma\, x_0<\cdots<x_n)$$
$$:=\sum(-1)^j f_\sigma^\tau(\lambda_\sigma)\, x_0<\cdots<\widehat{x}_j<\cdots<x_n$$



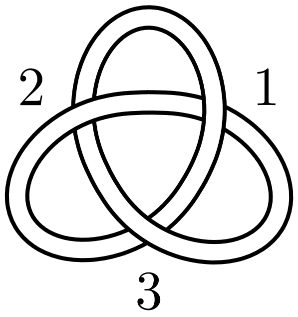
- homology $H_*(\Delta,\mathscr{A})$ with local coefficients $:=H_*(\mathcal{C}_*)$.

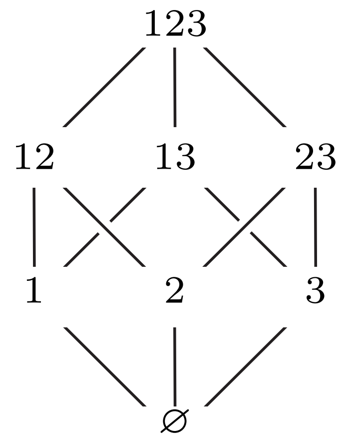
Coloured posets 1

- $(P, \mathfrak{F})$ — P: poset; $\mathfrak{F}$: functor $P \to R$-Mod

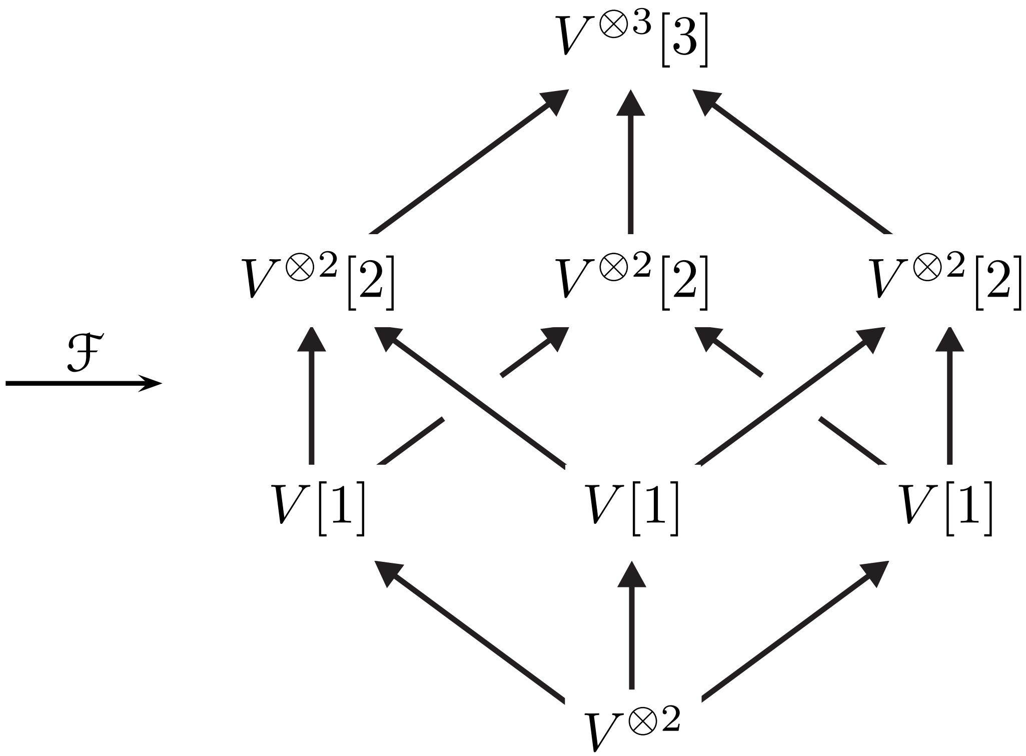


- Eg: Khovanov colouring of a Boolean lattice!





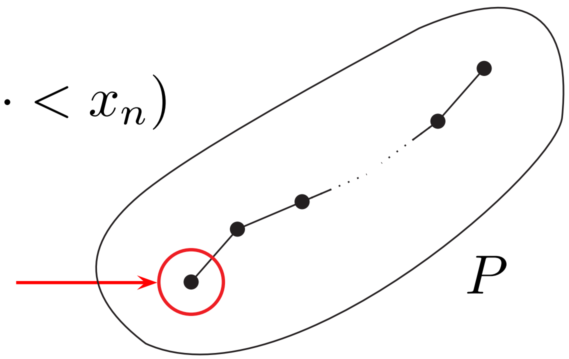
2^S, S = crossings



- coloured poset $(P, \mathfrak{F}) \to$ complex with local coeffs $(\Delta(P - \widehat{1}), \mathscr{A})$

n-cell $\sigma = (x_0 < \cdots < x_n)$

$A_\sigma = \mathfrak{F}(x_0)$



- order homology with local coeffs of coloured poset $(P, \mathfrak{F})$:

$$H_*(P, \mathfrak{F}) := H_*(\Delta(P - \widehat{1}), \mathscr{A})$$

- **Theorem** [E-T]: order homology with local coeffs of $(\text{Boolean}, \mathfrak{F} = \text{Khovanov}) \cong$ Khovanov homology.