Higher limits

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(joint with Paul Turner (Geneva))

1 | Higher limits

$P =$ small category

presheaf $F : P \to R\text{-mod}$ ($R = \mathbb{Z}$ in the examples in this §)

(contravariant)

$\varprojlim_P F =$ the $(a_x) \in \prod_P F(x)$

s.t.

$$\begin{tikzcd} & y \\ x \arrow[ur] \end{tikzcd} \in P \xrightarrow{\ F\ } \begin{tikzcd} & F(y) \arrow[dl] \\ F(x) \end{tikzcd}$$

gives $a_y \mapsto a_x$

Eg: $P = \circ \circlearrowleft^{g}$ $(g^2 = id)$

($\mathbb{Z}/2$, or any group)

$F = A \circlearrowleft g$

($\mathbb{Z}G$-module)

$\varprojlim (A \circlearrowleft g)$

$= \{a \in A \mid ga = a, \forall g\} = A^G$

Eg: $P =$
$$\begin{tikzcd} \circ & & \circ \\ \circ \arrow[u] \arrow[urr] & & \circ \arrow[u] \arrow[ull] \\ & \circ \arrow[ul] \arrow[ur] \end{tikzcd}$$
(or any poset)

$$\varprojlim \begin{tikzcd} A \arrow[d] \arrow[drr] & & 0 \arrow[d] \arrow[dll] \\ B \arrow[dr] & & C \arrow[dl] \\ & D \end{tikzcd} = \ker \begin{tikzcd} & A \arrow[dl] \arrow[dr] \\ B & \oplus & C \end{tikzcd}$$

<u>Higher limits</u> $\varprojlim^i_P F$ $(i \geq 0)$

$:= H^i(P, F)$

$=$ (right) derived functors of $\varprojlim_P$

$\varprojlim^i (A \circlearrowleft g)$

$= H^i(G, A)$

gp. cohomology

<u>to compute:</u> complex $S^*(P, F)$

$$\cdots \to S^{n-1} \xrightarrow{d} S^n \xrightarrow{d} S^{n+1} \to \cdots$$

$$S^n = \prod_\sigma F(x_n)$$

$(\sigma = x_n \to \cdots \to x_0)$

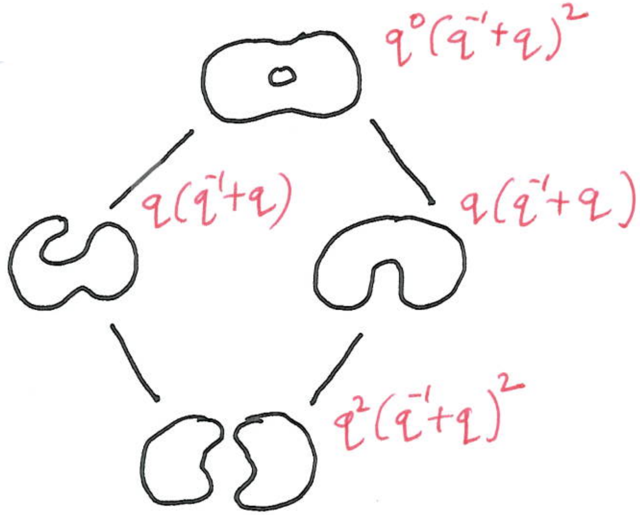
$$\varprojlim^i_P F = HS^i(P, F)$$

2 | Khovanov homology ($R = k$ a field)

$L =$ 

$$P = \begin{array}{ccc} & \phi & \\ 1 & & 2 \\ & 12 & \end{array}$$

$$\xrightarrow[x \mapsto q^{|x|}(q^{-1}+q)^{\# S^1}]{\text{Kauffman}}$$



$$\langle L \rangle = \sum (-1)^{|x|} \in \mathbb{Z}[q^{\pm 1}]$$

$$(= q^{-2} + q^0 + q^2 + q^4)$$

Khovanov $\downarrow$

$A = \underset{-1}{\mathbb{Q}} \oplus \underset{1}{\mathbb{Q}}$

$(\mathrm{rk} = q^{-1} + q)$

$x \mapsto A^{\otimes \# S^1}[|x|]$

Graded spaces: $A = \bigoplus_{\mathbb{Z}} A_i$

$\dim A = \sum \dim A_i \, q^i$

$\oplus \rightsquigarrow +$ dim's

$\otimes \rightsquigarrow \times$ dim's

$A[k] \rightsquigarrow q^k \dim A$

$$\begin{tikzcd} & A^{\otimes 2} \arrow[dl, "m"'] \arrow[dr, "m"] & & \arrow[r] & C^0 \arrow[d] \\ A[1] \arrow[dr, "\Delta"'] & & A[1] \arrow[dl, "-\Delta"] & \oplus \arrow[r] & C^1 \arrow[d] \\ & A^{\otimes 2}[2] & & \oplus \arrow[r] & C^2 \end{tikzcd}$$

(Khovanov presheaf)

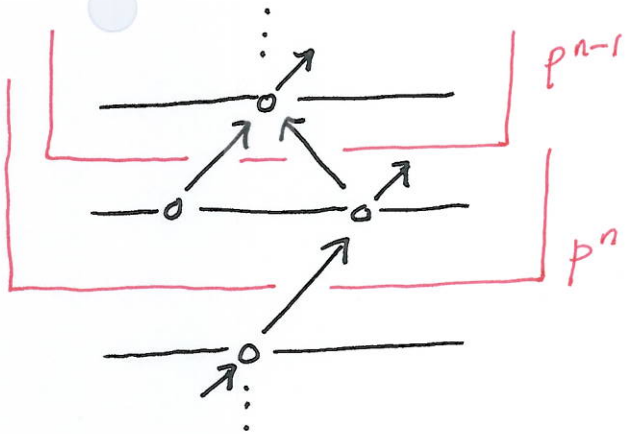
$Kh^i(L)$	dim
$\underset{-2}{\mathbb{Q}} \oplus \underset{0}{\mathbb{Q}}$	$q^{-2} + q^0$
0	0
$\underset{2}{\mathbb{Q}} \oplus \underset{4}{\mathbb{Q}}$	$q^2 + q^4$

Euler char. $\uparrow$

3 | Cellular homology

P = graded poset

F = presheaf on P



cellular complex $C^*(P,F)$

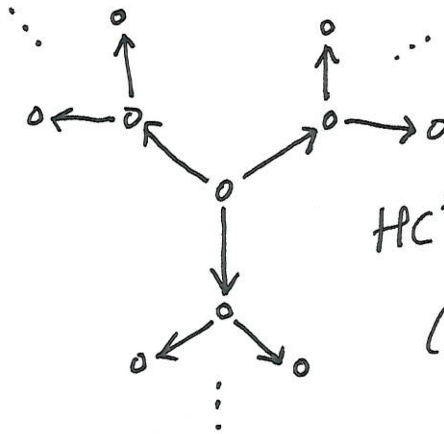
$$C^n = HS^n(P^n, P^{n-1}; F)$$

Theorem (E.-Turner)

P graded, locally finite, cellular

$$\varprojlim_P{}^i F = HC^i(P,F)$$

(non-)Eg:



$$HC^*(P,F) \neq \varprojlim_P{}^* F$$

(any F)

(+ P finite + $(R=k)$ + $\dim_k C^i < \infty$)

Cor (categorification formula) P as above

$$\sum_i (-1)^i \dim \varprojlim_P{}^i F = -\sum_x \mu_x \dim F(x)$$

$$(\mu_x = \mu(\underline{x}, \bot))$$

Eg 1: P = subsets of crossings of L

F = Khovanov presheaf

$\overset{\text{Th}^{\text{m}}}{\Longrightarrow}$ $\varprojlim_P{}^i F = Kh^i(L)$ Khovanov

$\overset{\text{Cor.}}{\Longrightarrow}$ $\sum (-1)^i \dim \varprojlim_P{}^i F = \langle L \rangle$ Kauffman

Eg 2: G/B flag variety, P = Schubert varieties

red. gp. / $\mathbb{C}$ (G), Borel (B)

($\cong$ Weyl gp. Bruhat $\leq$)

presheaf $F(x) = \bigoplus_{x \leq y} \mathcal{H}^*(y)$ intersection homology

$$\overset{\text{Cor.}}{\Longrightarrow} \sum (-1)^i \dim \varprojlim_P{}^i F = \begin{cases} P_{KL} & \ell(w_0) \equiv 0\ (2) \\ 2 - P_{KL} & \ell(w_0) \equiv 1\ (2) \end{cases}$$