Partial symmetry, reflection monoids and Coxeter groups

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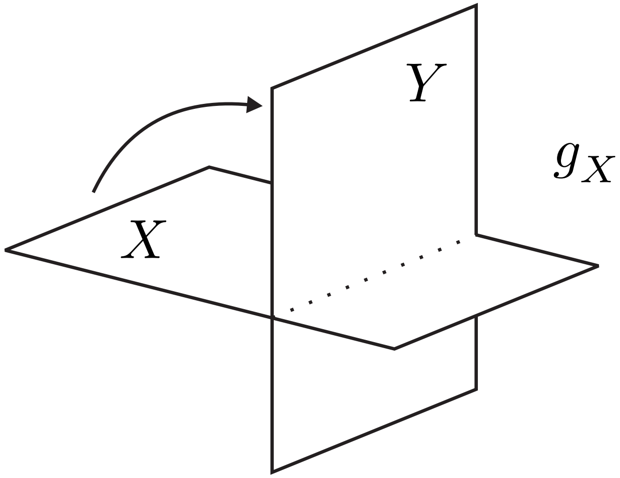
- V finite dimensional space over $\mathbb{F}$; $GL(V)$ = group of isomorphisms $V \xrightarrow{g} V$.
- partial (linear) isomorphism of V: vector space isomorphism $X \to Y$ for $X, Y \subset V$ subspaces:

$$g_X := \begin{cases} g \in GL(V) \text{ on } X, \\ \text{undefined elsewhere.} \end{cases}$$



$$g_X f_Z = g f_{X \cap Z g^{-1}}$$

- $ML(V)$:= general linear monoid.
- $G \subset GL(V)$; $\mathcal{S}$ system of subspaces for G: $\overset{\text{def}}{\Leftrightarrow} \begin{cases} V \in \mathcal{S}, \\ X, Y \in \mathcal{S} \Rightarrow X \cap Y \in \mathcal{S}, \\ \mathcal{S}G = \mathcal{S}. \end{cases}$
- monoid of partial linear isomorphisms $M(G, \mathcal{S}) := \{g_X \,|\, g \in G, X \in \mathcal{S}\}$.

• Eg: for any $G \subset GL(V)$ there is a "natural" choice of system, namely the subspaces,

$$\text{Fix}(H) = \{\mathbf{v} \in V \,|\, (\mathbf{v})g = \mathbf{v} \text{ for all } g \in H\},$$

for all $H \subset G$ subgroups.

Theorem 1 *Let $G \subset GL(V)$ be a finite group, $\mathcal{S}$ a finite system in V for G, and $M(G, \mathcal{S})$ the resulting monoid of partial linear isomorphisms. Then*

$$|M(G, \mathcal{S})| = \sum_{X \in \mathcal{S}} [G : G_X]$$

where G_X is the isotropy group of $X \in \mathcal{S}$:

$$G_X = \{g \in G \,|\, (\mathbf{v})g = \mathbf{v}, \text{ for all } \mathbf{v} \in X\}.$$

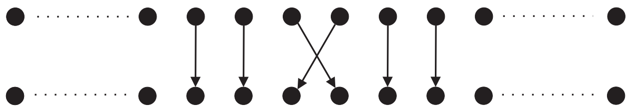
- V Euclidean space with orthonormal basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$.
- $\mathfrak{S}_n \subset GL(V)$ via $\mathbf{x}_i\sigma = \mathbf{x}_{i\sigma}$ (ie: permutation matrices).
- For $J \subset I = \{1, \ldots, n\}$, let

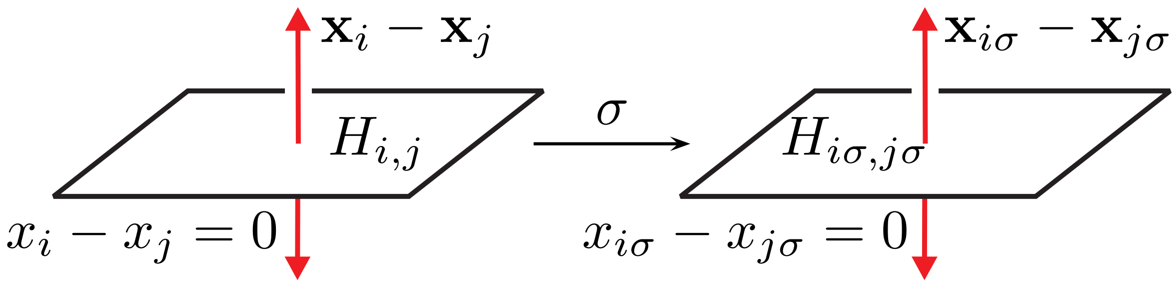
$$X(J) = \bigoplus_{j \in J} \mathbb{R}\mathbf{x}_j \subset V,$$

so that $X(I) = V$, $X(J)\sigma = X(J\sigma)$, and $X(J_1) \bigcap X(J_2) = X(J_1 \cap J_2)$.
$\Rightarrow \mathcal{B} = \{X(J) \mid J \subset I\}$ the Boolean system for $\mathfrak{S}_n$.

- Isotropy group of $X(J)$ is $\cong \mathfrak{S}_{I \setminus J}$, so Theorem 1 $\Rightarrow$

$$|M(\mathfrak{S}_n, \mathcal{B})| = \sum_{J \subset I} [\mathfrak{S}_I : \mathfrak{S}_{I \setminus J}] = |\mathfrak{S}_n| \sum_{k=0}^{n} \binom{n}{k} \frac{1}{|\mathfrak{S}_{n-k}|} = \sum_{k=0}^{n} \binom{n}{k}^2 k!$$

- $M(\mathfrak{S}_n, \mathcal{B}) \cong \mathcal{I}_n$  $\rightarrow g_{X(J)}$ for $g = (i, i+1)$.



- V and $\mathfrak{S}_n \subset GL(V)$ as before.
- $\mathcal{A} = \{H_{i,j} \,|\, 1 \leq i \neq j \leq n\}$

$\mathcal{H} = \left\{ \bigcap_{H \in \mathcal{X}} H \,|\, \mathcal{X} \subset \mathcal{A} \right\}$ Coxeter arrangement system for $\mathfrak{S}_n$.

- $|M(\mathfrak{S}_n, \mathcal{H})| = \sum_{\Lambda} [\mathfrak{S}_I : \mathfrak{S}_{\Lambda_1} \times \cdots \times \mathfrak{S}_{\Lambda_p}] = (n!)^2 \sum_{\lambda} \frac{1}{b_\lambda \lambda_1! \ldots \lambda_p!}$

($\Lambda = \{\Lambda_1, \ldots, \Lambda_p\}$ partition of I; $\lambda_i = |\Lambda_i|$; $\lambda = (\lambda_1, \ldots, \lambda_p)$ partition of n; $b_\lambda = b_1! \ldots b_n! (1!)^{b_1} \ldots (n!)^{b_n}$, b_j = number of $\lambda_i = j$)

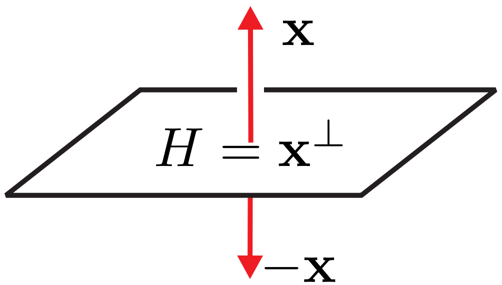
- ($M(\mathfrak{S}_n, \mathcal{H}) \cong$ monoid of uniform block permutations.)

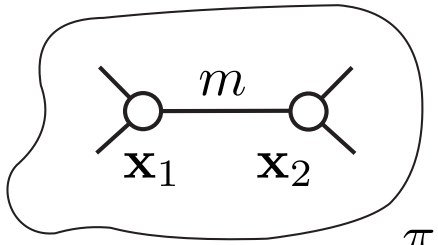
• reflection $V \xrightarrow{s\neq\text{id}} V \in GL(V)$ semisimple, fixes hyperplane pointwise.
reflection group $W := \langle\text{reflections } s \in S\rangle \subset GL(V)$.

• finite reflection groups classified when $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{F}_q, \mathbb{Q}_p$.

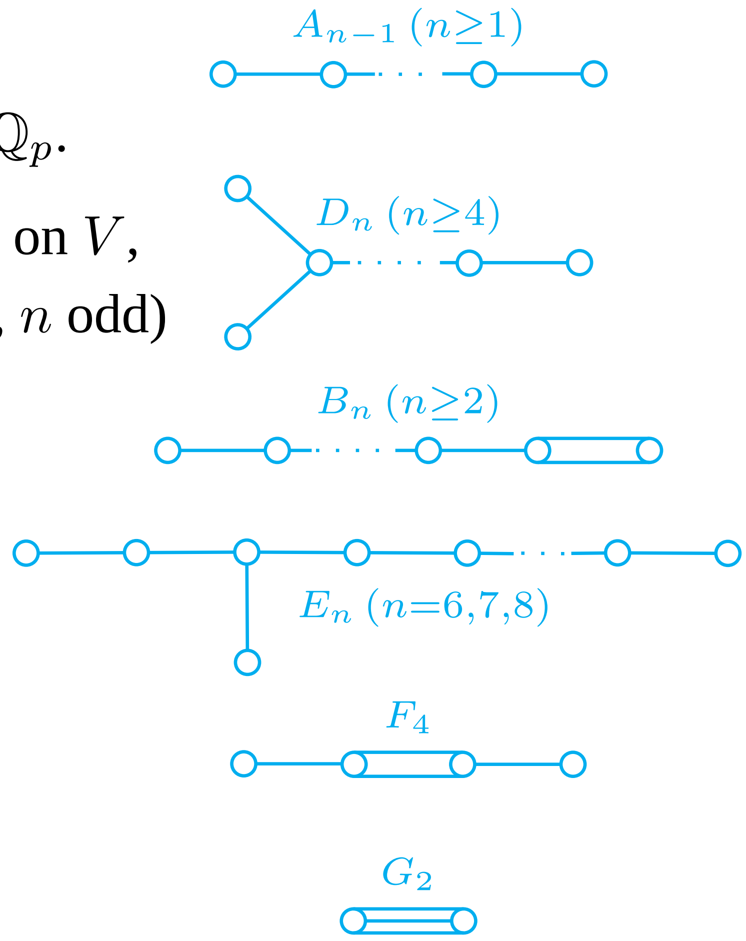
• Eg: $\mathbb{F} = \mathbb{F}_q$ (q odd), Q non-degenerate quadratic form on V,
$O(V,Q)$ = orthogonal group ($O_n^{\pm}(q)$, n even, or $O_n^{\circ}(q)$, n odd)

• finite $\mathbb{R}$ groups:





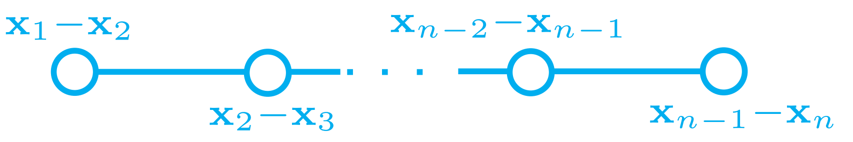
$$(\mathbf{x}_1, \mathbf{x}_2) = -\cos\frac{\pi}{m}$$

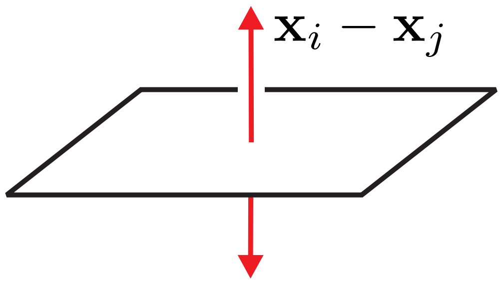


• $\mathbb{F} = \mathbb{C}, \mathbb{F}_q$: W finite $\Rightarrow W_X$ a reflection group.
($\mathbb{F} = \mathbb{C}$ (Steinberg), $\mathbb{F} = \mathbb{F}_q$ (Nakajima-Serre))

• . . . are the $M(W, \mathcal{S})$ for $W \subset GL(V)$ a reflection group.
($\Leftrightarrow$ factorizable inverse submonoids of $ML(V)$ generated by partial reflections).

• Eg:



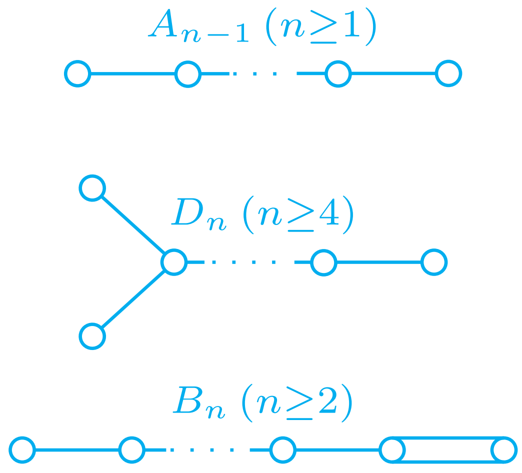


Identifies $W(A_{n-1})$ with $\mathfrak{S}_n \subset GL(V)$ acting as before.

$\Rightarrow$ two Eg's earlier are reflection monoids

(write $M(A_{n-1}, \mathcal{B})$, $M(A_{n-1}, \mathcal{H})$ . . .)

• ie: $W(A_{n-1}) \cong \mathfrak{S}_n$.
$M(A_{n-1}, \mathcal{B}) \cong \mathcal{I}_n$.



• $\mathcal{B}=\{X(J)=\bigoplus_{j\in J}\mathbb{R}\mathbf{x}_j \mid J\subset I=\{1,\ldots,n\}\}$

system for W in types $\Phi=A, B$ and D

$M(\Phi,\mathcal{B})$ the Boolean reflection monoids

Theorem 2 $\Phi=A_{n-1}, B_n$ *or* D_n,

$$|M(\Phi,\mathcal{B})|=\sum_{J\subset I}[W(\Phi):W(\Phi_{I\setminus J})]=|W(\Phi_n)|\sum_{k=0}^{n}\binom{n}{k}\frac{1}{|W(\Phi_{n-k})|}.$$

$(\Phi_T=\Phi\cap X(T))$

• Eg: $W(B_n)\cong$ group of signed permutations;
$M(B_n,\mathcal{B})\cong$ monoid of partial signed permutations.

- $W \subset GL(V)$ finite (real) reflection group.

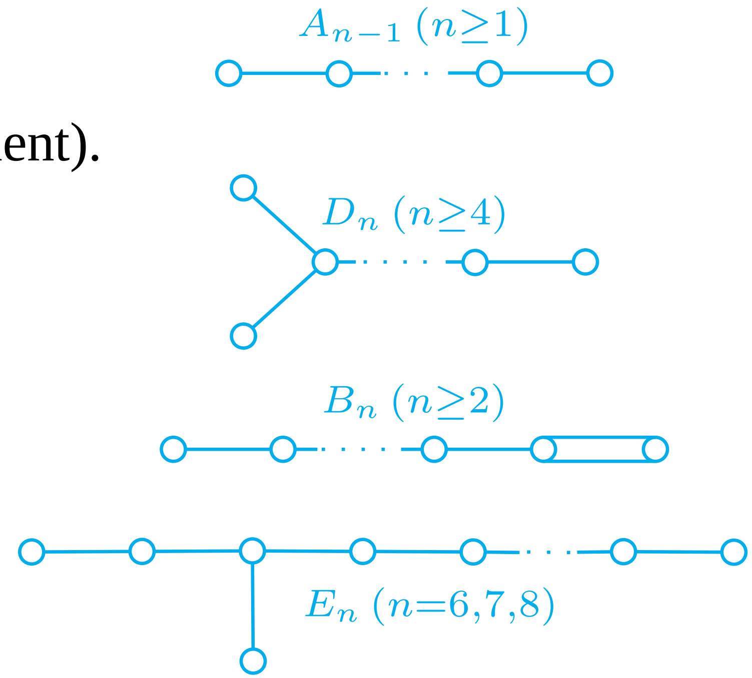
$\mathcal{A}$ = reflecting hyperplanes of W (hyperplane arrangement).

$\mathcal{H} = \left\{ \bigcap_{H \in \mathcal{X}} H \mid \mathcal{X} \subset \mathcal{A} \right\}$ Coxeter arrangement system.

($\mathcal{H}$ = intersection lattice of arrangement $\mathcal{A}$)

$\mathcal{H}$ = the "natural" system $\{\text{Fix}(H) \mid H \subset W\}$ for W.

$M(W, \mathcal{H})$ Coxeter arrangement monoid.

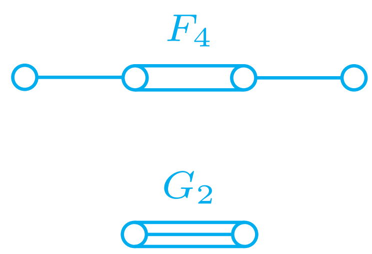


Theorem 3 $|M(W, \mathcal{H})| = \sum [W : W']$, *sum over parabolic subgroups* $W' \subset W$.

- Eg: $|M(B_n, \mathcal{H})| = 2^{2n-1}(n!)^2 \sum_{m,\lambda} \frac{1}{4^m d_\lambda}$

the sum over all $0 \leq m \leq n$ and partitions λ of $n - m$.

$(d_\lambda = 4^p b_\lambda \lambda_1! \ldots \lambda_p!)$



- algebraic monoid $\mathbb{M} :=$ $\begin{cases} \text{variety over } k + \text{monoid} \\ \text{Zariski closed submonoid of } \mathbf{M}_n \end{cases}$
- algebraic group $\mathbb{G} := \mathbb{M} \cap \mathbf{GL}_n$

$\mathbb{M} \subset \mathbf{M}_n$

$\cap \uparrow \downarrow -$

$\mathbb{G} \subset GL_n$

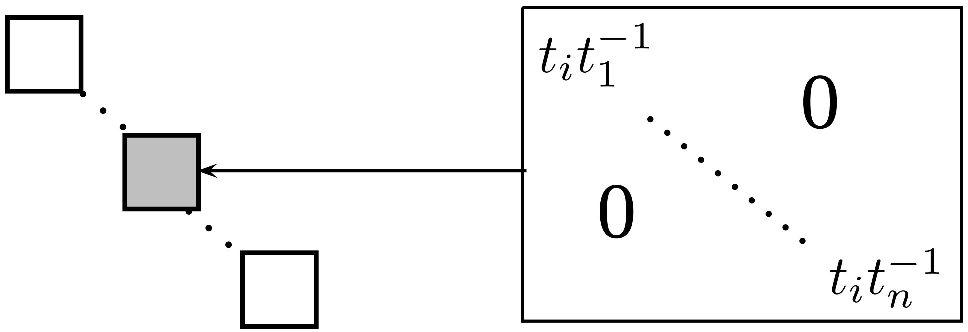
- Eg: $V = n$-dimensional k-space

$\rho : \mathbf{SL}_n \to GL(V \otimes V)$ with $g(v \otimes v') = gv \otimes (g^{-1})^T v'$

$\mathbb{G} = k^* \rho(\mathbf{SL}_n) \subset GL_{n^2}$ and $\mathbb{M} = \overline{\mathbb{G}}$

$[\mathbb{G} = k^* \rho(\mathbb{G}_0), \mathbb{M} = \overline{G}]$

$\rho\,\text{diag}(t_1, \ldots, t_n) =$



$[T = k^* \rho T_0$ maximal torus$]$

$\mathfrak{X}(\rho T_0) := \text{Hom}(\rho T_0, k^*)$ spanned by $\chi_{ij}(g) = t_i t_j^{-1}$ with $\chi_{ji} = \chi_{ij}^{-1}$, $\chi_{ij} = \chi_{i\ell} \chi_{j\ell}^{-1}$

$\mathfrak{X}(\rho T_0)$ free Abelian rank $n - 1 \Rightarrow V := \mathfrak{X}(\rho T_0) \otimes \mathbb{R}$ $(n-1)$-dim real space

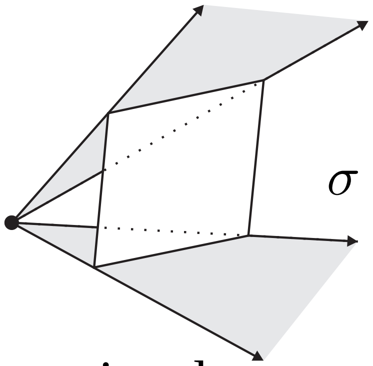
$\mathfrak{S}_n$ acts on V via $\chi_{ij}\sigma = \chi_{i\sigma, j\sigma}$

$[W_{\mathbb{G}}$=Weyl group$]$

- understanding $\mathbb{M}$ (units $=\mathbb{G}$, idempotents $=E$):

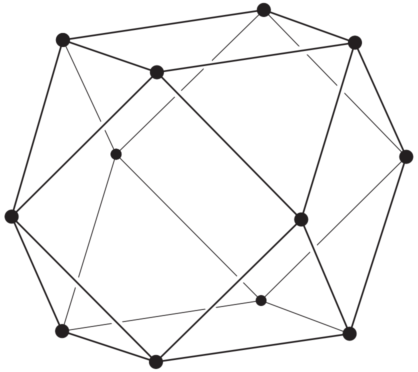
$$W_{\mathbb{G}} \circlearrowright \mathfrak{X}(T)\otimes\mathbb{R}\Big\}\,{\color{red}\mathbb{G}\circlearrowright E}\Big\{E(\overline{T})\circlearrowleft W_{\mathbb{G}}\Big\{\text{cone } \sigma\subset\mathfrak{X}(T)\otimes\mathbb{R}\circlearrowleft W_{\mathbb{G}}$$

- $E(\overline{T}) \xrightarrow{\cong}$ face lattice $\mathcal{F}(\sigma)$



- Eg ($n=4$): $V=\mathfrak{X}(\rho T_0)\otimes\mathbb{R}$ 3-dimensional

$\sigma=$ cone on:



• $W_{\mathbb{G}}$-action on cone $\sigma \Rightarrow \mathcal{S}_{\mathbb{M}} := \left\{ \bigcap_{\tau \in \mathcal{X}} \mathbb{R}\tau \mid \mathcal{X} \subset \mathcal{F}(\sigma) \right\}$ a system for $W_{\mathbb{G}}$. $M(W_{\mathbb{G}}, \mathcal{S}_{\mathbb{M}})$ the reflection monoid associated to $\mathbb{M}$.

• Renner monoid $R_{\mathbb{M}}$ parametrises the "Bruhat decomposition" of $\mathbb{M}$.

Theorem 4 *($\mathbb{M}$ connected reductive with 0) There is a surjective homomorphism*

$$f : M(W_{\mathbb{G}}, \mathcal{B}_{\mathbb{M}}) \to R_{\mathbb{M}}$$

which is injective if and only if σ is a simplicial cone.