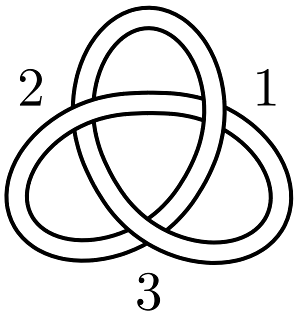
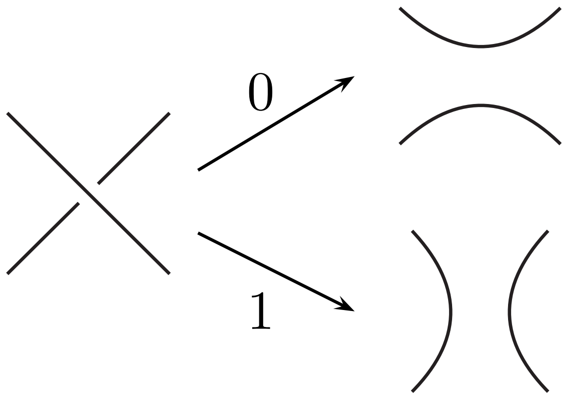
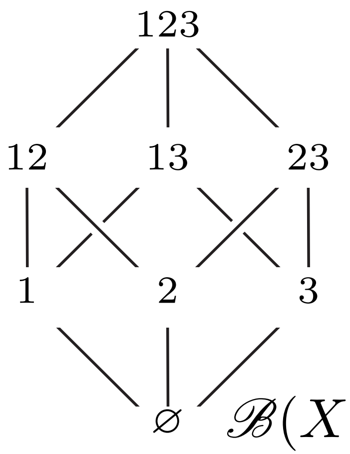
Coloured poset homology

Brent Everitt (York) –joint with Paul Turner (Fribourg)

arXiv:0808.1686
arXiv:0711.0103 (in *J. Algebra*)





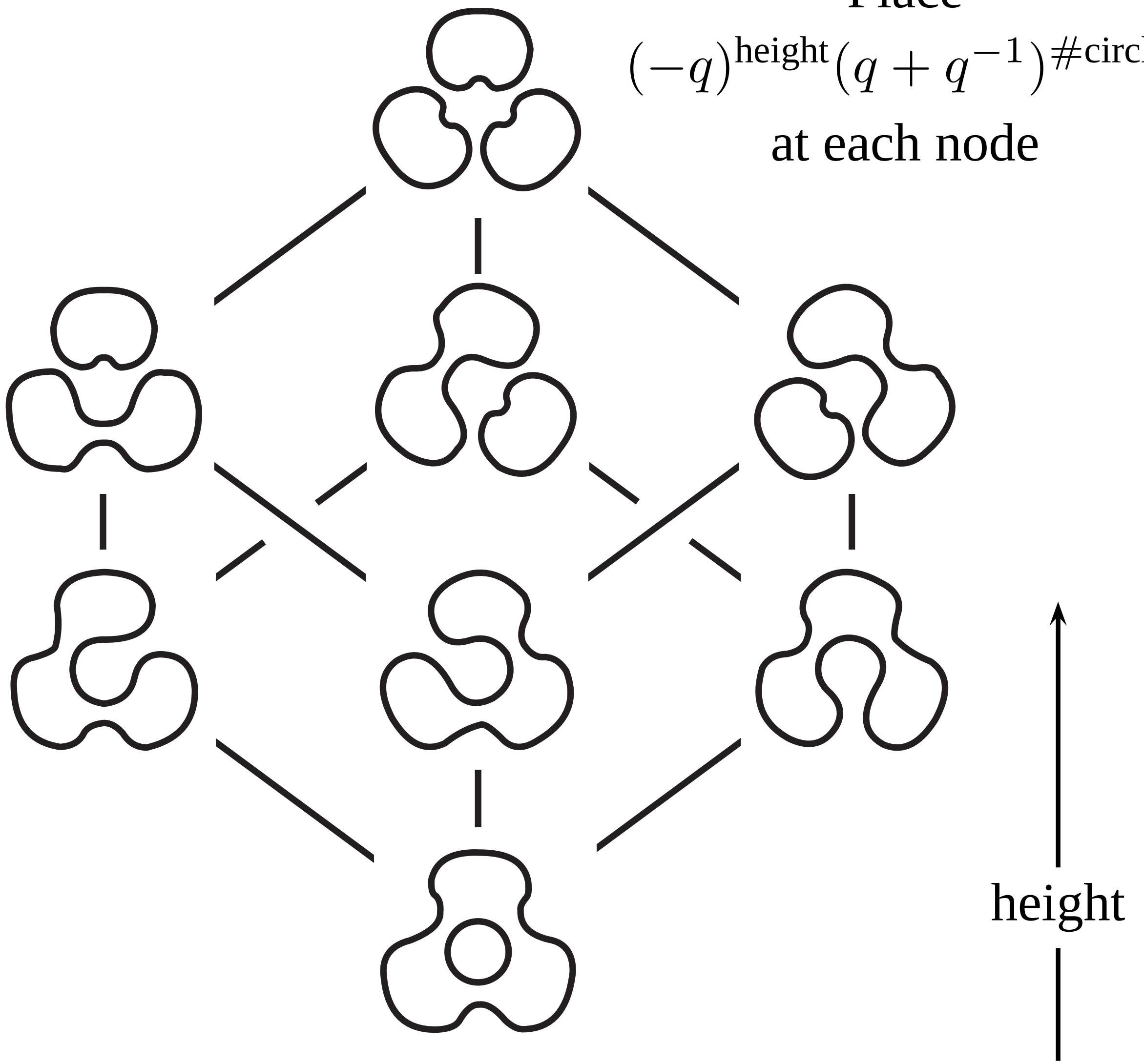


$\mathscr{B}(X), X = \{1, 2, 3\}$

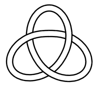
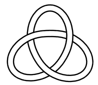
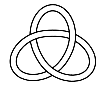
Place

$(-q)^{\text{height}}(q + q^{-1})^{\#\text{circles}}$

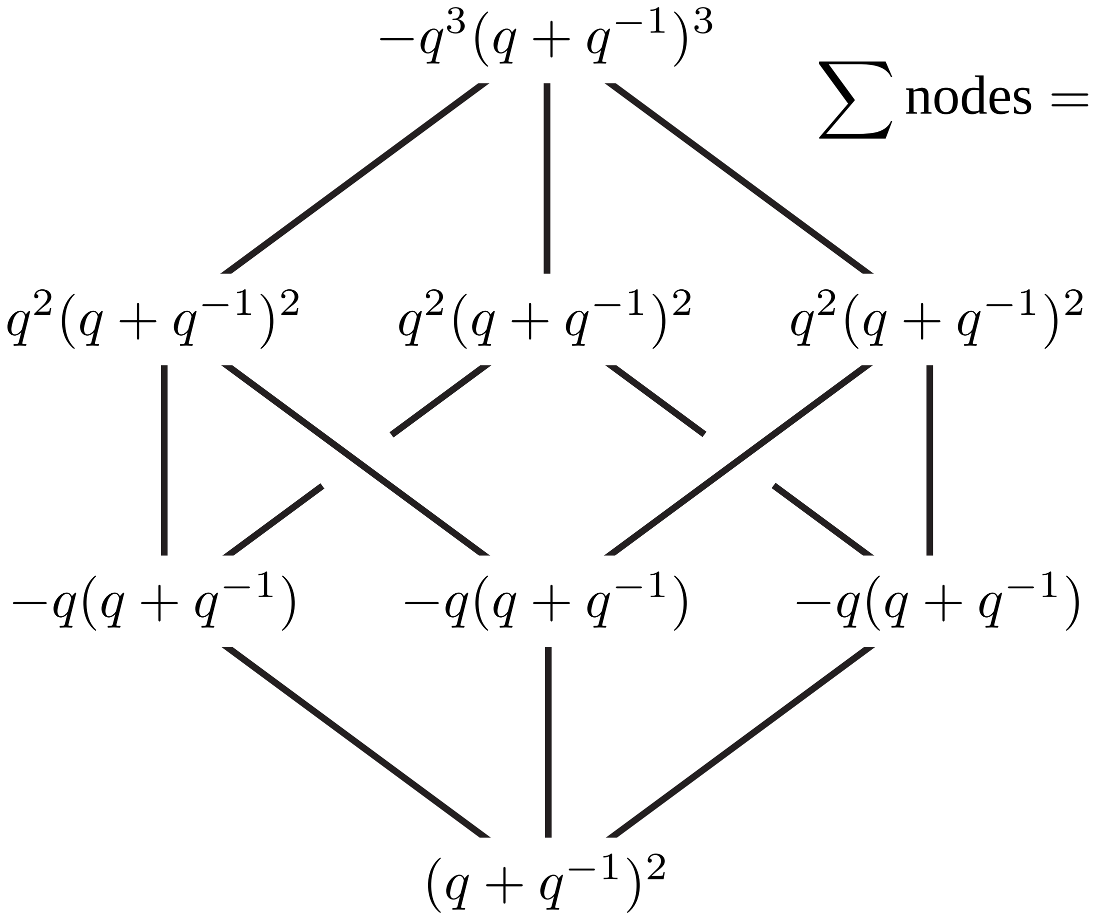
at each node

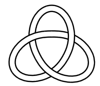


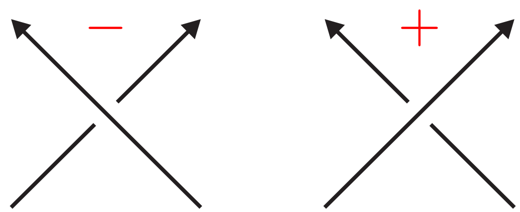
Jones polynomial

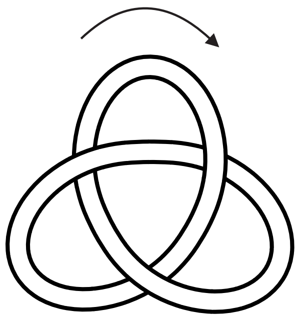
$J\Big($  $\Big) = \dfrac{1}{(q+q^{-1})}\widehat{J}\Big($  $\Big) \longleftarrow (-1)^{n_-} q^{n_+ - 2n_-} \Big\langle$  $\Big\rangle$

(Jones) (unnormalized Jones) (Kauffman bracket)



$\sum$ nodes $= \Big\langle$  $\Big\rangle = -q^6 + q^2 + 1 + q^{-2}$





$\mathbb{N} = \{0, 1, 2, \ldots\}$ —categorify→ **Spaces**

←de-categorify $(\dim)$—

Obj: k-spaces V

Mor: k-linear maps

$\mathbb{N}[q, q^{-1}] = \{\sum_{\mathbb{Z}} n_i q^i\}$ —categorify→ **Graded Spaces**

←de-categorify $(q\dim)$—

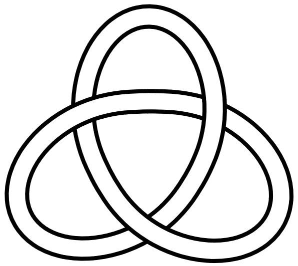
Obj: $V =$

$\cdots$	V_{-1}	V_0	V_1	$\cdots$

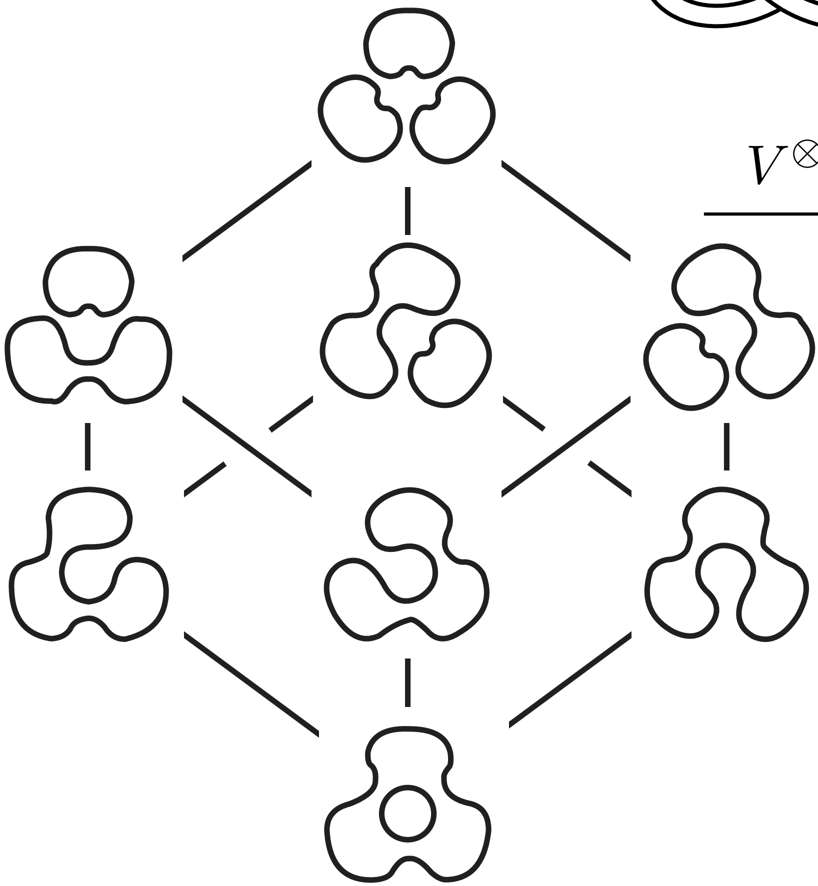
Mor: graded maps

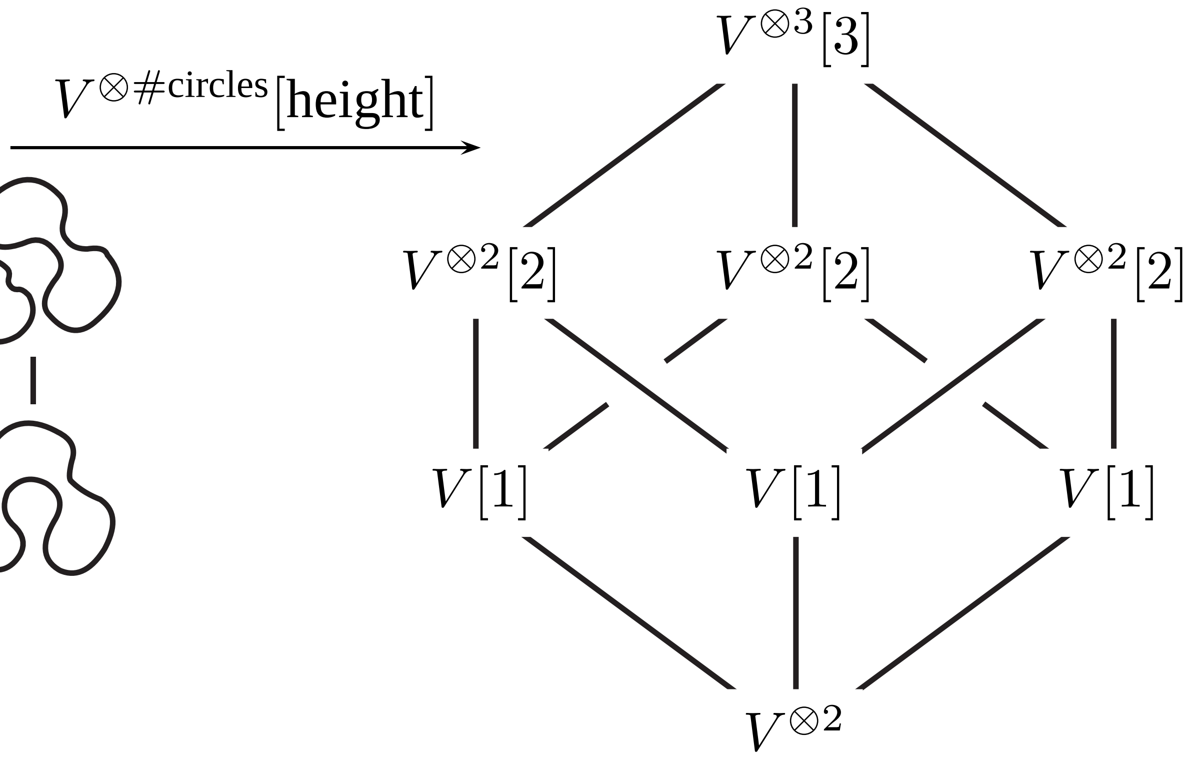
$q\dim V = \sum \dim V_i q^i$

Khovanov complex

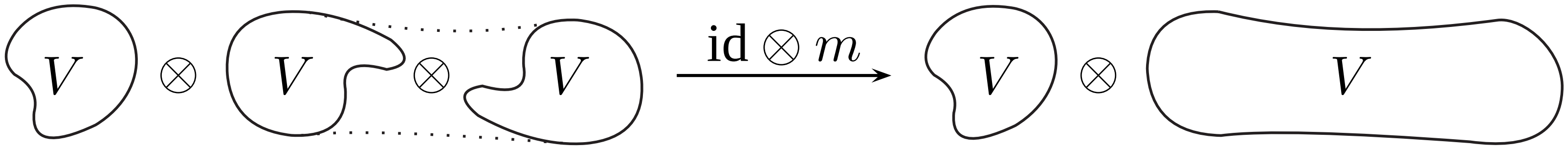


$$V = \mathbb{Q} \oplus \mathbb{Q}$$
$$\quad\quad\;\; {\color{red}-1} \quad {\color{red}1}$$

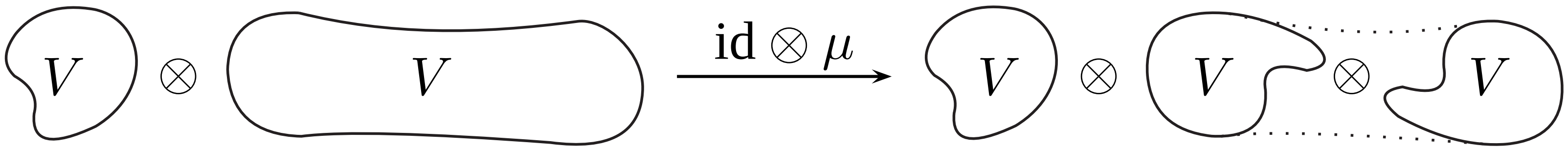


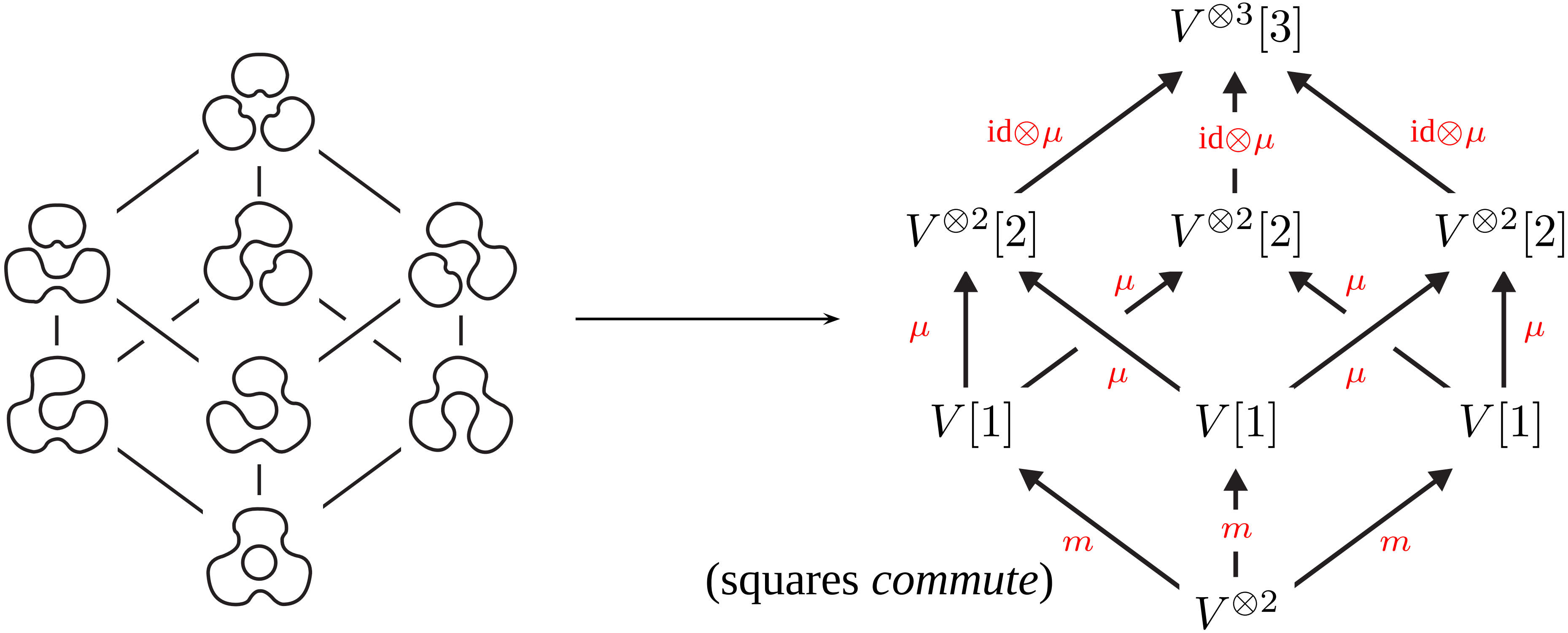


$m : V \otimes V \to V$

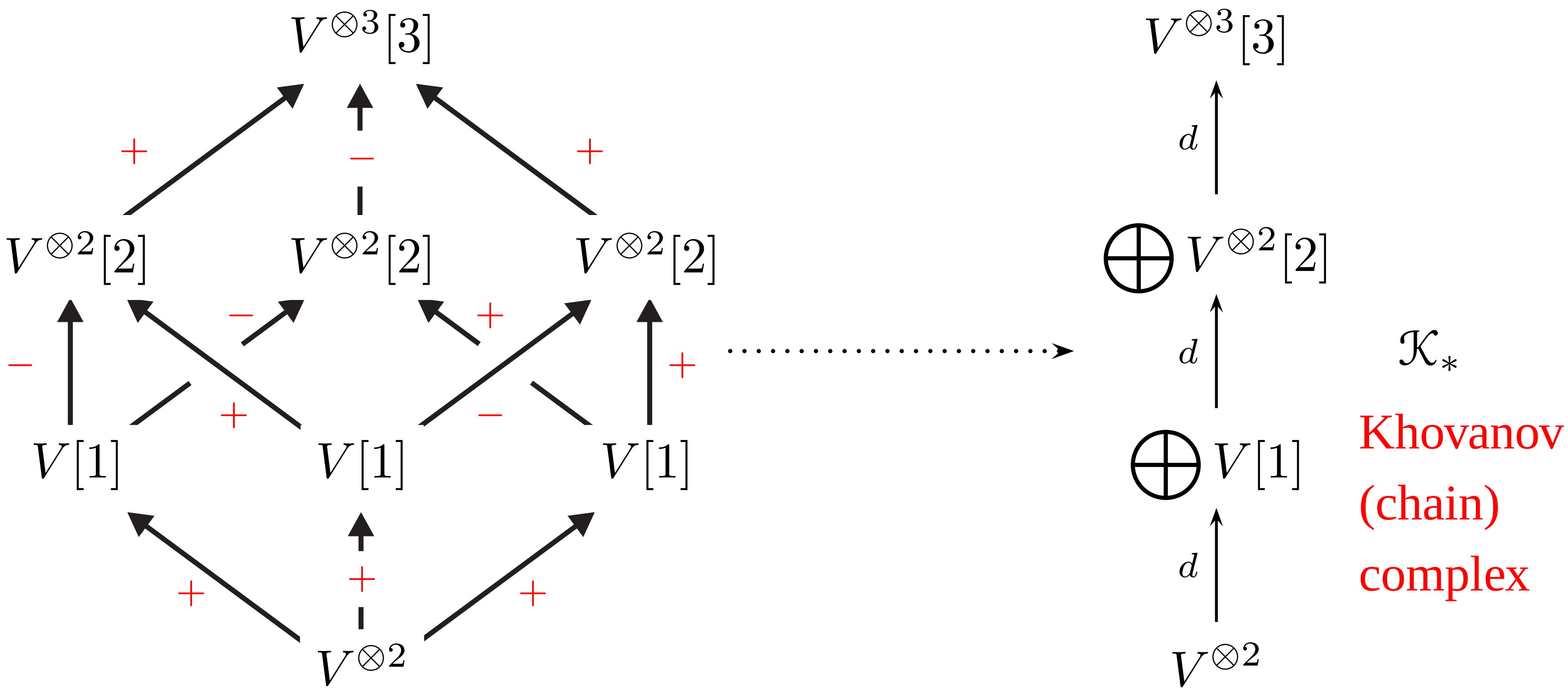


$\mu : V \to V \otimes V$

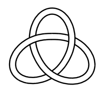


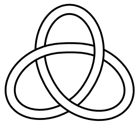


(squares *commute*)

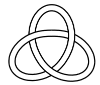


add $\pm$'s to edge maps so squares *anticommute*

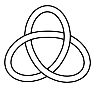
Khovanov homology $KH_*\Big($  $, \mathbb{Q}\Big) = H_*(\mathcal{K}_*)$

	6	4	2	0	-2	$q\mathrm{dim}$
KH_0	$\mathbb{Q}$					q^6
KH_1			$\mathbb{Q}$			q^2
KH_2						0
KH_3				$\mathbb{Q}$	$\mathbb{Q}$	$1+q^{-2}$

Euler characteristic $\chi(\mathcal{K}_*)$

$= \sum(-1)^i q\mathrm{dim}\, KH_i \Big($  $, \mathbb{Q}\Big)$

$= q^6 - q^2 - 1 - q^{-2}$

Jones $\Big($  $\Big) \in \mathbb{Z}[q, q^{-1}]$ — categorify → bi-Graded Spaces

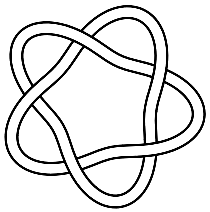
← de-categorify (χ)

$$\chi(V) = \sum(-1)^i q\mathrm{dim}\, KH_i$$

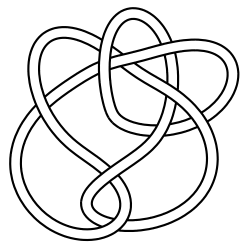
$\cdots$	$V_{i,-1}$	$V_{i,0}$	$V_{i,1}$	$\cdots$

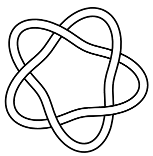
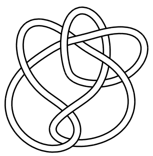
KH_{i-1}
KH_i
KH_{i+1}

$\mathbb{Q}$						
		$\mathbb{Q}$				
		$\mathbb{Q}$				
				$\mathbb{Q}$		
					$\mathbb{Q}$	$\mathbb{Q}$



$\mathbb{Q}$							
		$\mathbb{Q}$					
		$\mathbb{Q}$					
			$\mathbb{Q}$	$\mathbb{Q}$			
			$\mathbb{Q}$		$\mathbb{Q}$		
					$\mathbb{Q}\oplus\mathbb{Q}$		
							$\mathbb{Q}$
						$\mathbb{Q}$	$\mathbb{Q}$



Jones () = Jones ()

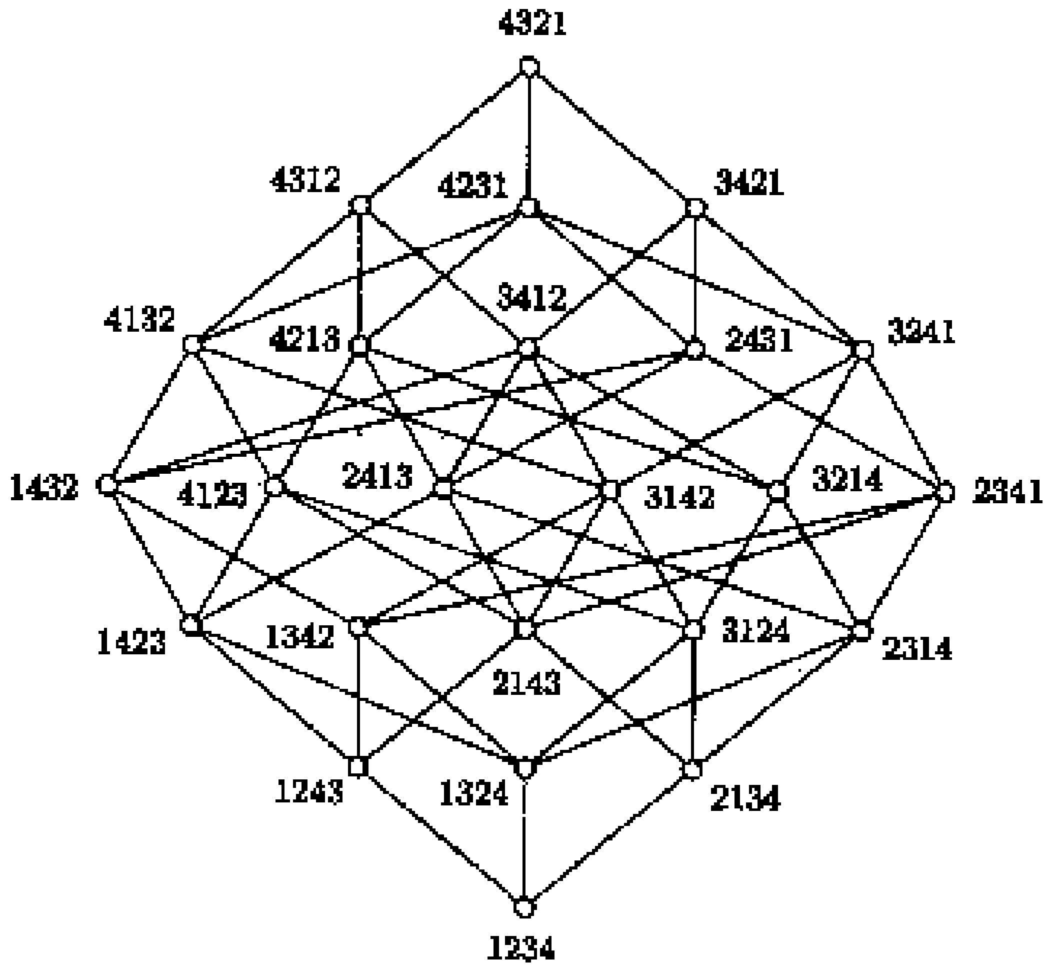
Eg:

- Boolean lattice $\mathscr{B}(X)$ on a set X.

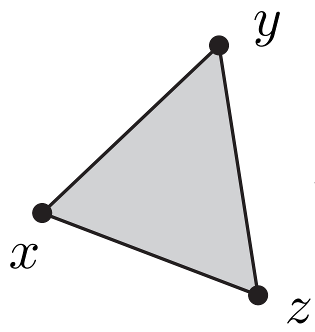
- Coxeter group with the Bruhat-Chevelley order, Eg: $\mathfrak{S}_4$ →

- cell poset of a CW-complex.

- intersection lattice of a hyper-plane arrangement.



- poset $P \longrightarrow |P|$ order (simplicial) complex.



iff $x < y < z$

- poset homology = simplicial homology of $|P|$

ie: $H_*(P, R) := H_*(|P|, R) =$ homology of chain complex

$$C_n(P, R) = \bigoplus_{x_0 < \cdots < x_n} R$$

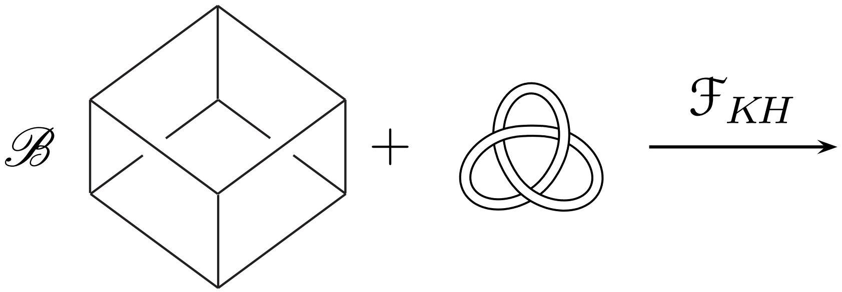
with differential $d : C_n(P, R) \to C_{n-1}(P, R)$

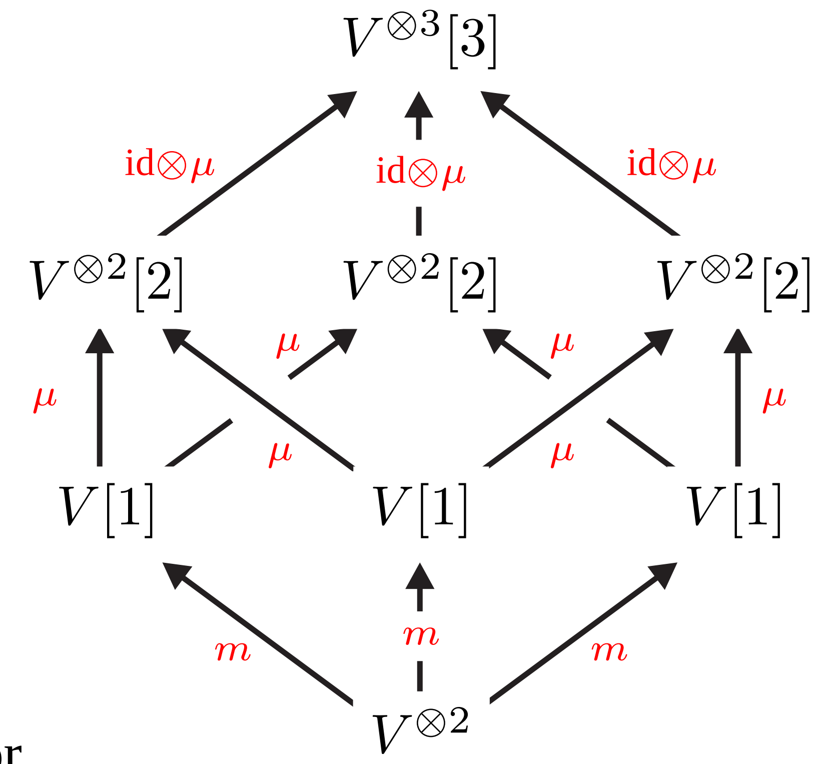
$$\lambda \cdot (x_0 < \cdots < x_n) \overset{d}{\mapsto} \sum_{j=0}^{n} (-1)^j \lambda \cdot (x_0 < \cdots < \widehat{x}_j < \cdots < x_n)$$

- Eg: [Folkman] P finite geometric lattice

$$\widetilde{H}_n(P \setminus \{0,1\}, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{|\mu(0,1)|} & n = \mathrm{rk}P - 2, \\ 0 & \text{otherwise.} \end{cases}$$

- Eg: “Khovanov colouring”:





- in general: $P \xrightarrow{\mathfrak{F}} R$-**mod** (covariant) functor

(= pre-sheaf of modules over P)

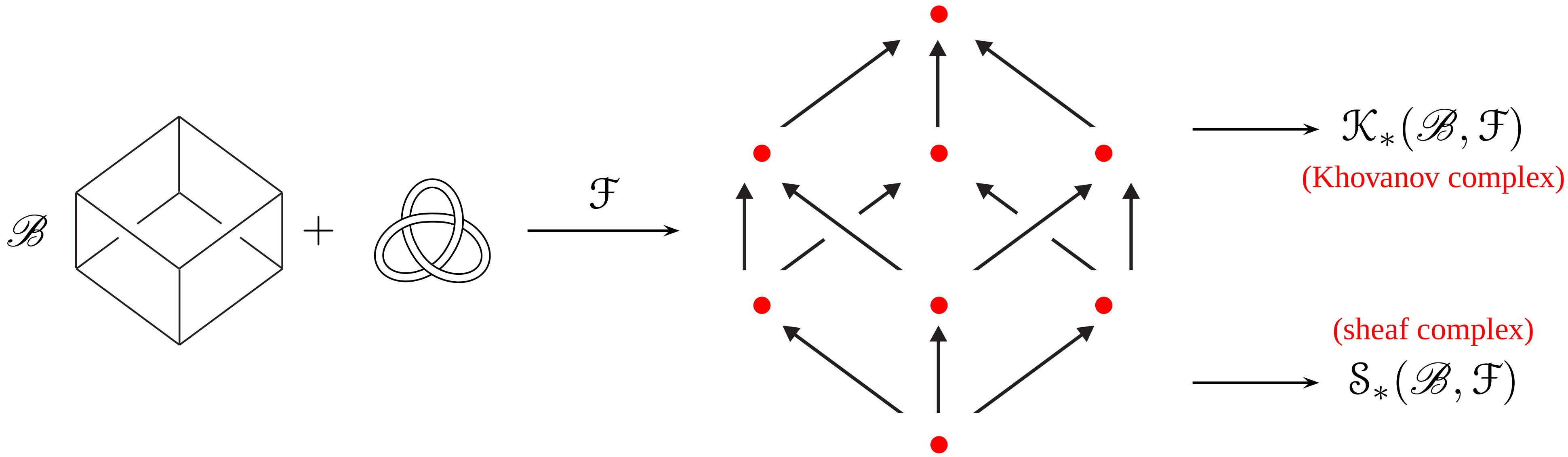
- $P \xrightarrow{\mathcal{F}} R$-**mod** coloured poset/sheaf

- coloured poset or sheaf homology $\mathcal{H}_*(P, \mathcal{F})$ = homology of chain complex

$$\mathcal{S}_n(P, \mathcal{F}) = \bigoplus_{x_0 < \cdots < x_n} \mathcal{F}(x_0)$$

with differential $d : \mathcal{S}_n(P, \mathcal{F}) \to \mathcal{S}_{n-1}(P, \mathcal{F})$

$$\lambda \cdot (x_0 < \cdots < x_n) \stackrel{d}{\mapsto} \mathcal{F}(x_0 < x_1)(\lambda) \cdot (\widehat{x}_0 < x_1 < \cdots < x_n) + \sum_{j=1}^{n} (-1)^j \lambda \cdot (x_0 < \cdots < \widehat{x}_j < \cdots < x_n)$$



- **Theorem**: $\mathscr{B}$ Boolean and $\mathscr{B} \stackrel{\mathcal{F}}{\to} R$-**mod** a sheaf, then

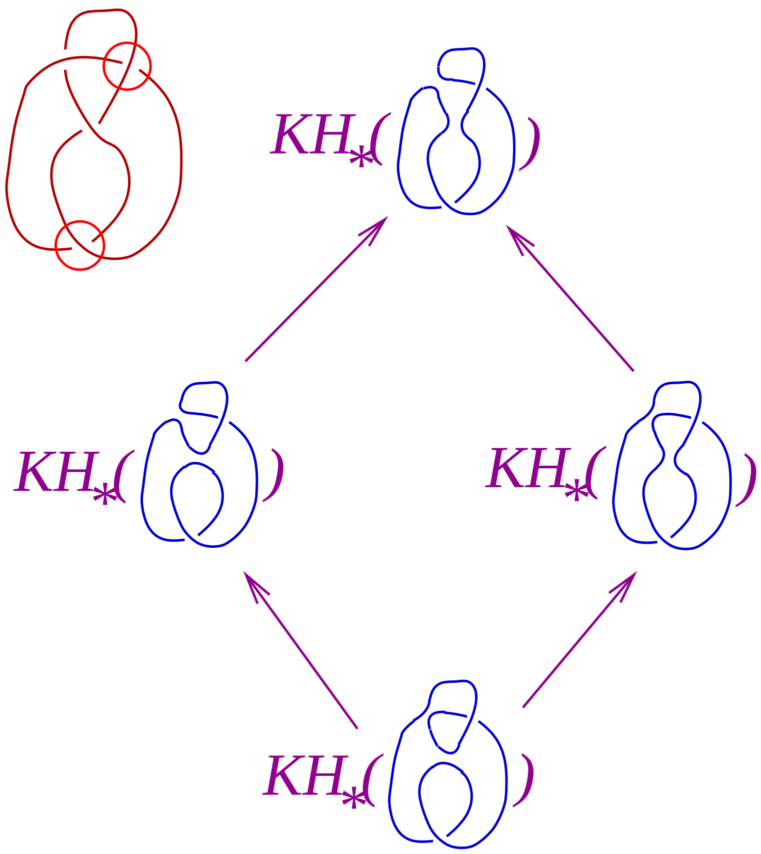
$$KH_*(\mathscr{B},\mathcal{F}) \cong \widetilde{\mathcal{H}}_{*-1}(\mathscr{B}\setminus 1,\mathcal{F})$$

- **Theorem**: P cellular poset and $P \stackrel{\mathcal{F}}{\to} R$-**mod** a sheaf, then

$$H_*^{\text{cell}}(P,\mathcal{F}) \cong \mathcal{H}_*(P,\mathcal{F})$$

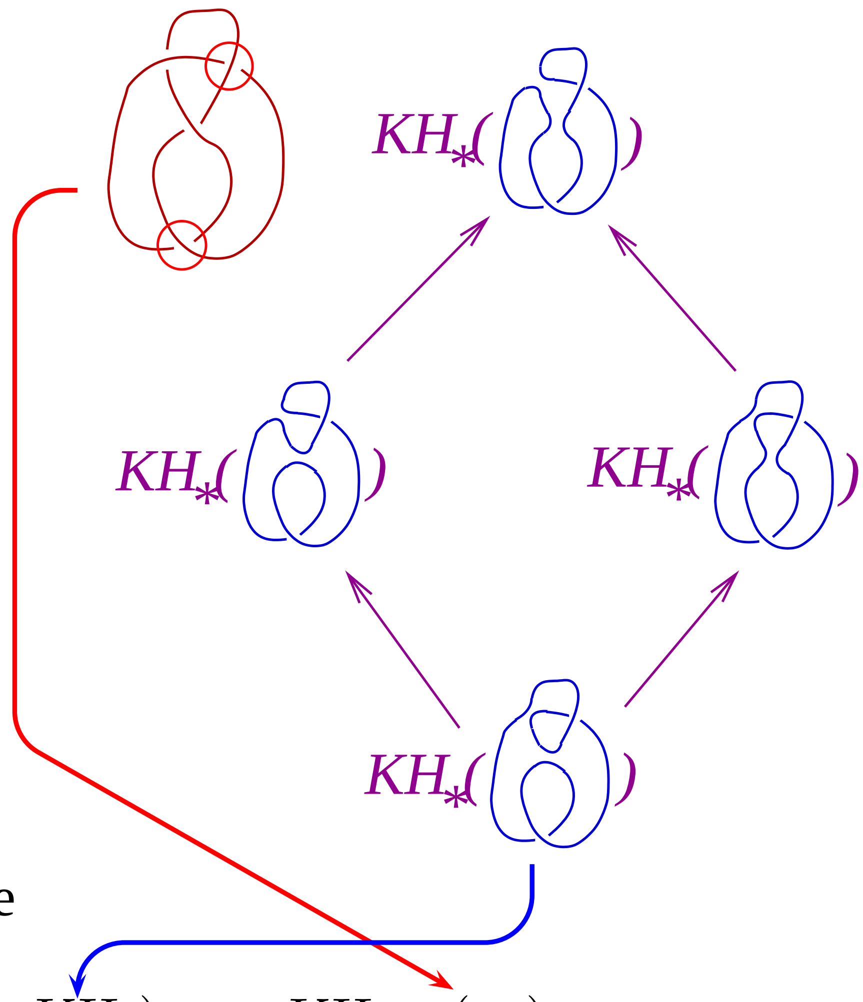
(Eg: Cohen-Macaulay posets, cell posets of regular CW-complexes,. . .)

- Take an N-crossing link diagram D and fix k crossings.
- Resolve each of the remaining crossings as usual.
- Put the resulting 2^{N-k} diagrams on a Boolean lattice $\mathscr{B}$.
- Define a sheaf on $\mathscr{B}$ by taking $KH_*(-)$.



Application: bundles of sheaves

- Take an N-crossing link diagram D and fix k crossings.
- Resolve each of the remaining crossings as usual.
- Put the resulting 2^{N-k} diagrams on a Boolean lattice $\mathscr{B}$.
- Define a sheaf on $\mathscr{B}$ by taking $KH_*(-)$.



Theorem: There is a spectral sequence

$$E^2_{p,q} = KH_p(\mathscr{B}, KH_q) \Longrightarrow KH_{p+q}(\quad)$$