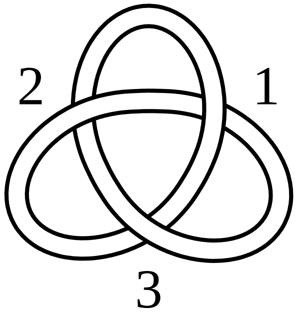
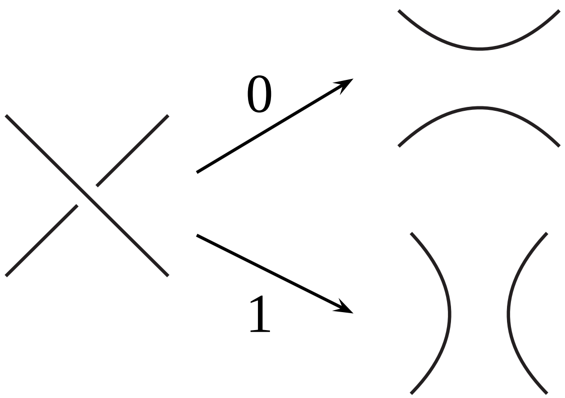
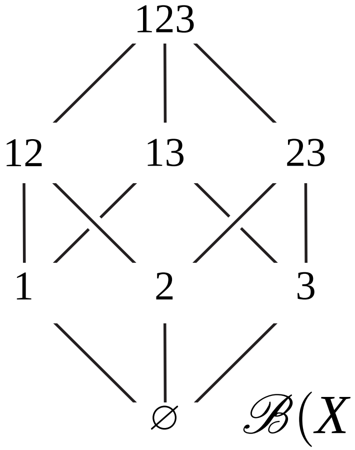
Knots, posets and sheaves

Brent Everitt (York) –joint with Paul Turner (Fribourg)





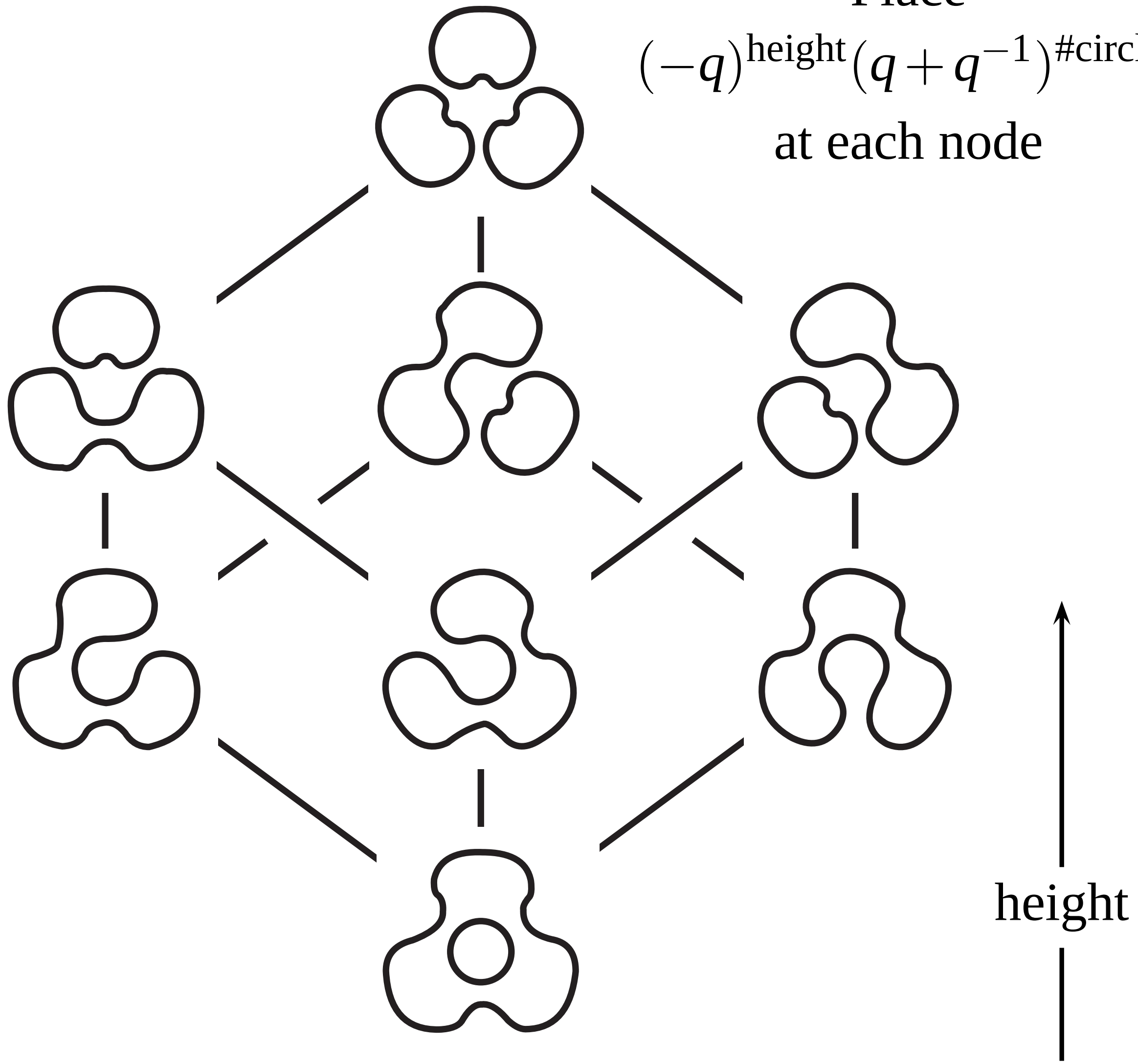


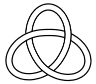
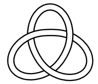
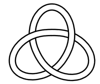
$\mathscr{B}(X), X = \{1,2,3\}$

Place

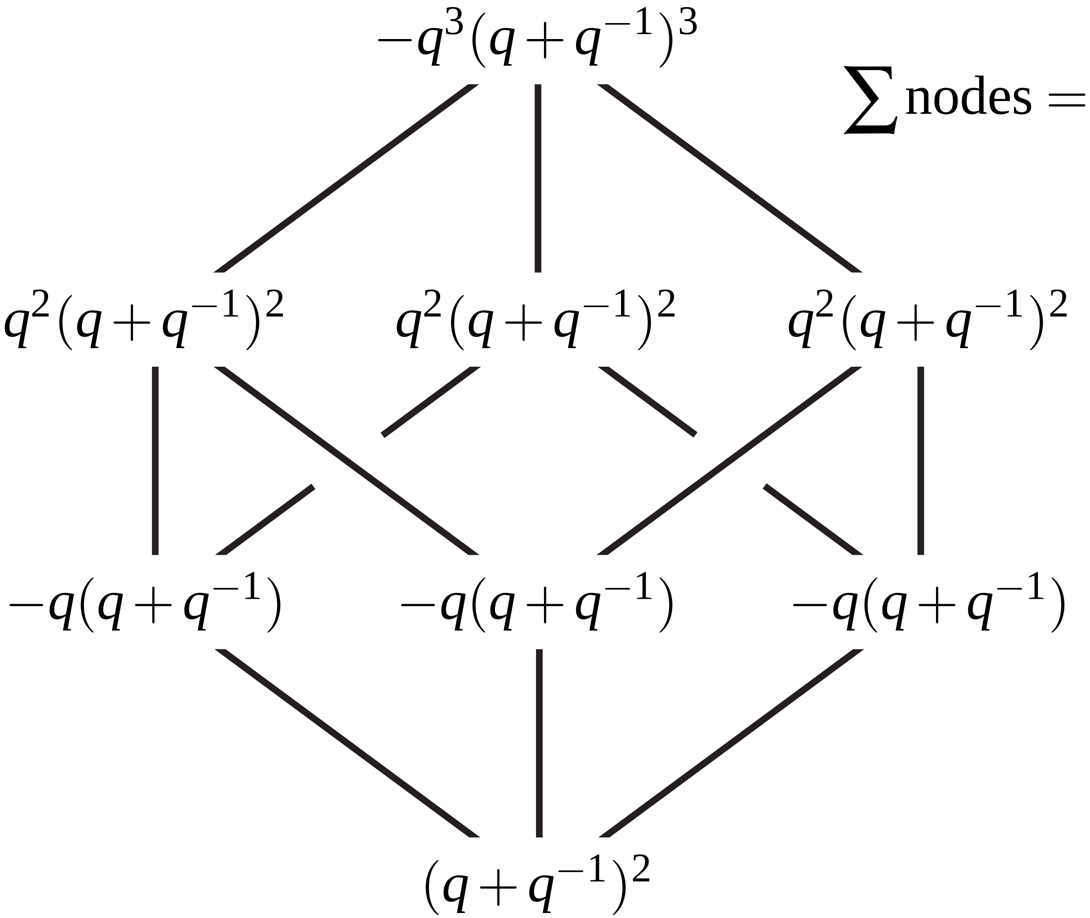
$(-q)^{\text{height}}(q+q^{-1})^{\#\text{circles}}$

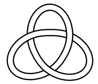
at each node

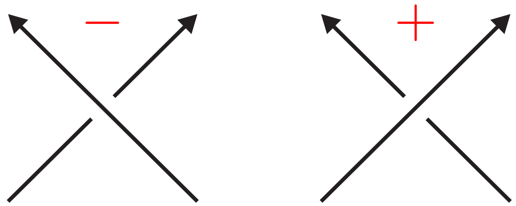


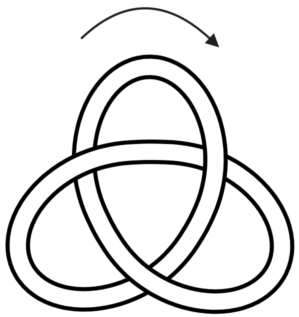
$J\Big($  $\Big) = \dfrac{1}{(q+q^{-1})}\widehat{J}\Big($  $\Big) \longleftarrow (-1)^{n_-}q^{n_+-2n_-}\Big\langle$  $\Big\rangle$

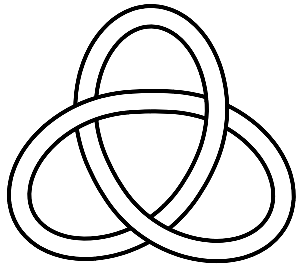
(Jones) (unnormalized Jones) (Kauffman bracket)



$\sum \text{nodes} = \Big\langle$  $\Big\rangle = -q^6+q^2+1+q^{-2}$

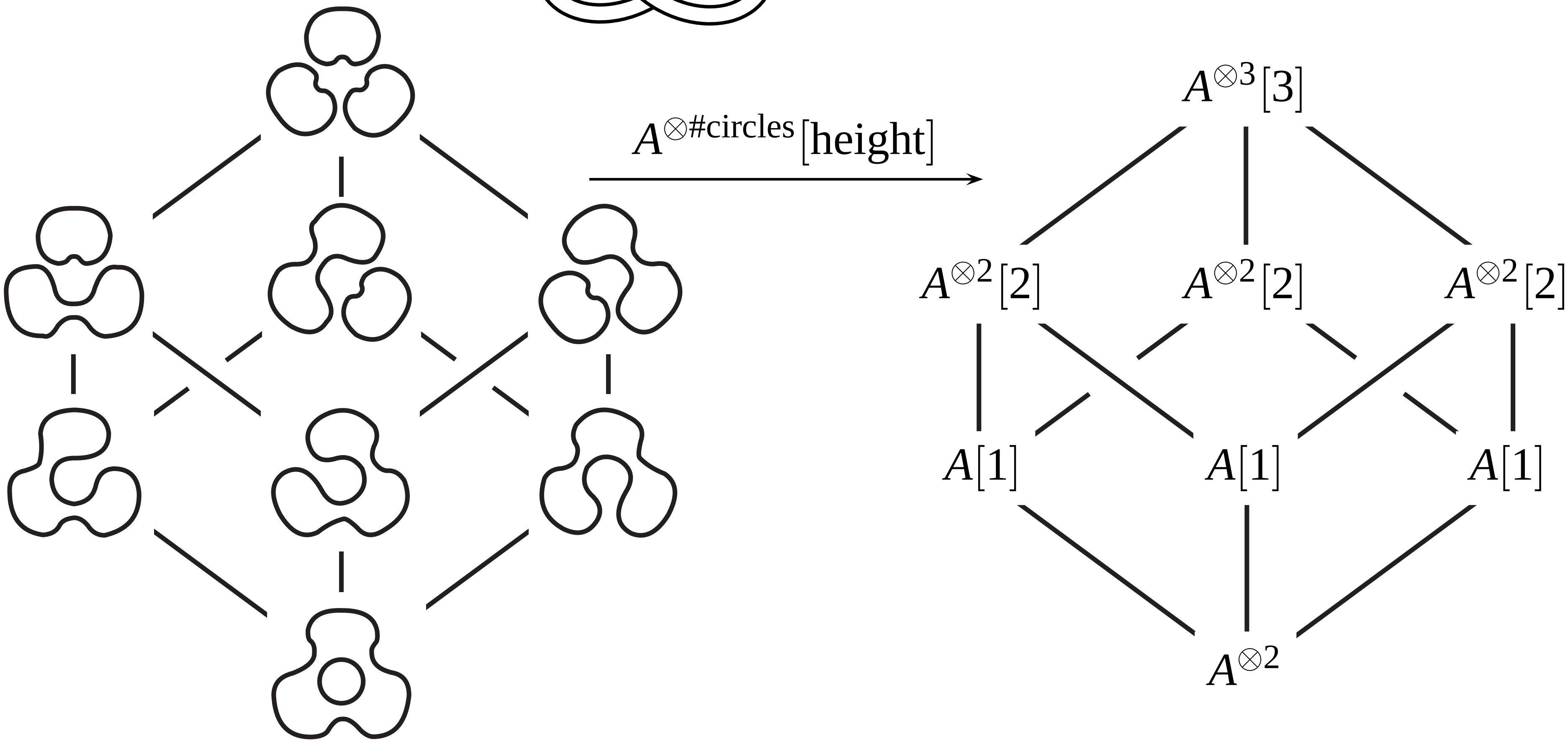




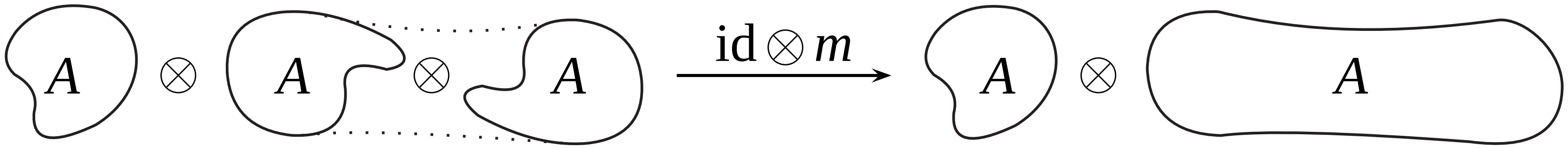


$A = \mathbb{Q} \oplus \mathbb{Q}$

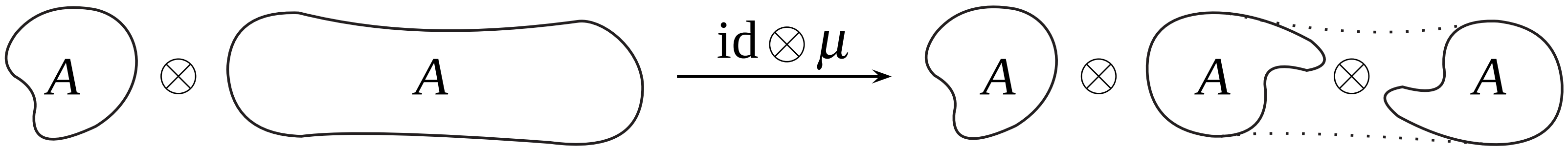
$-1 \quad 1$

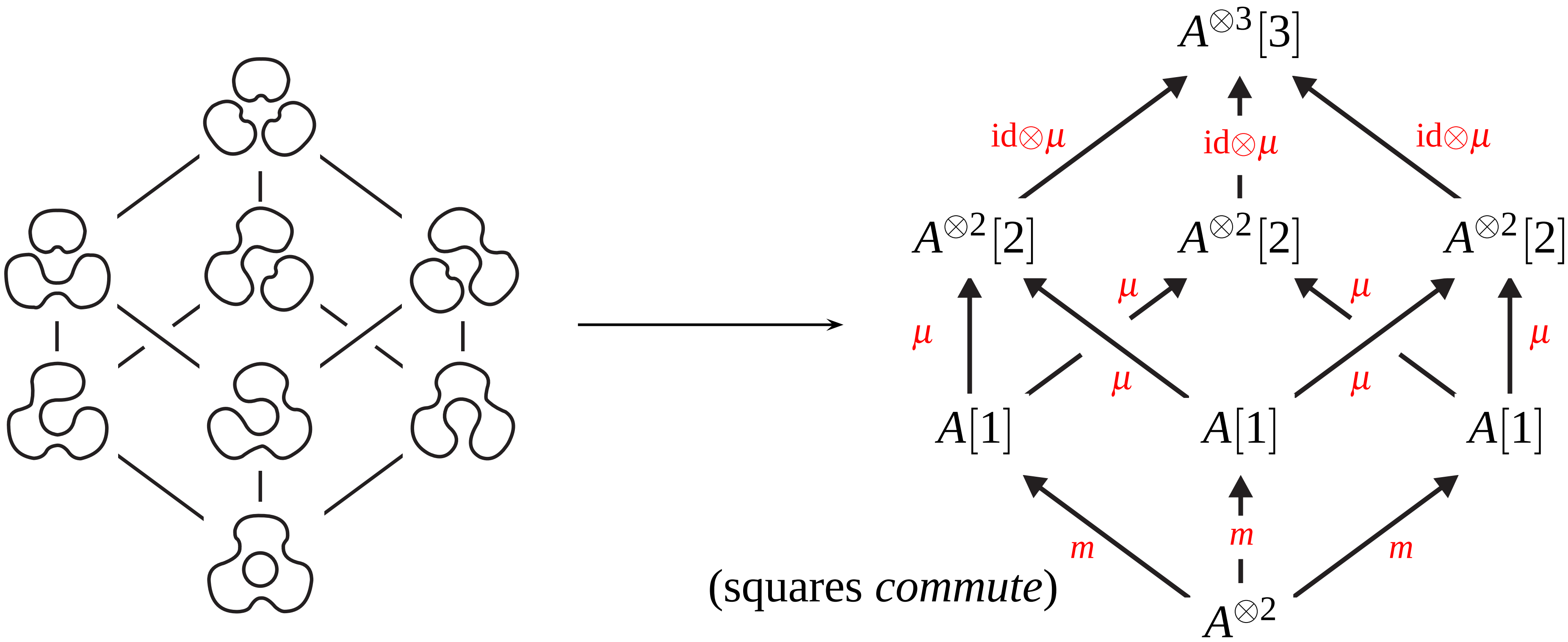


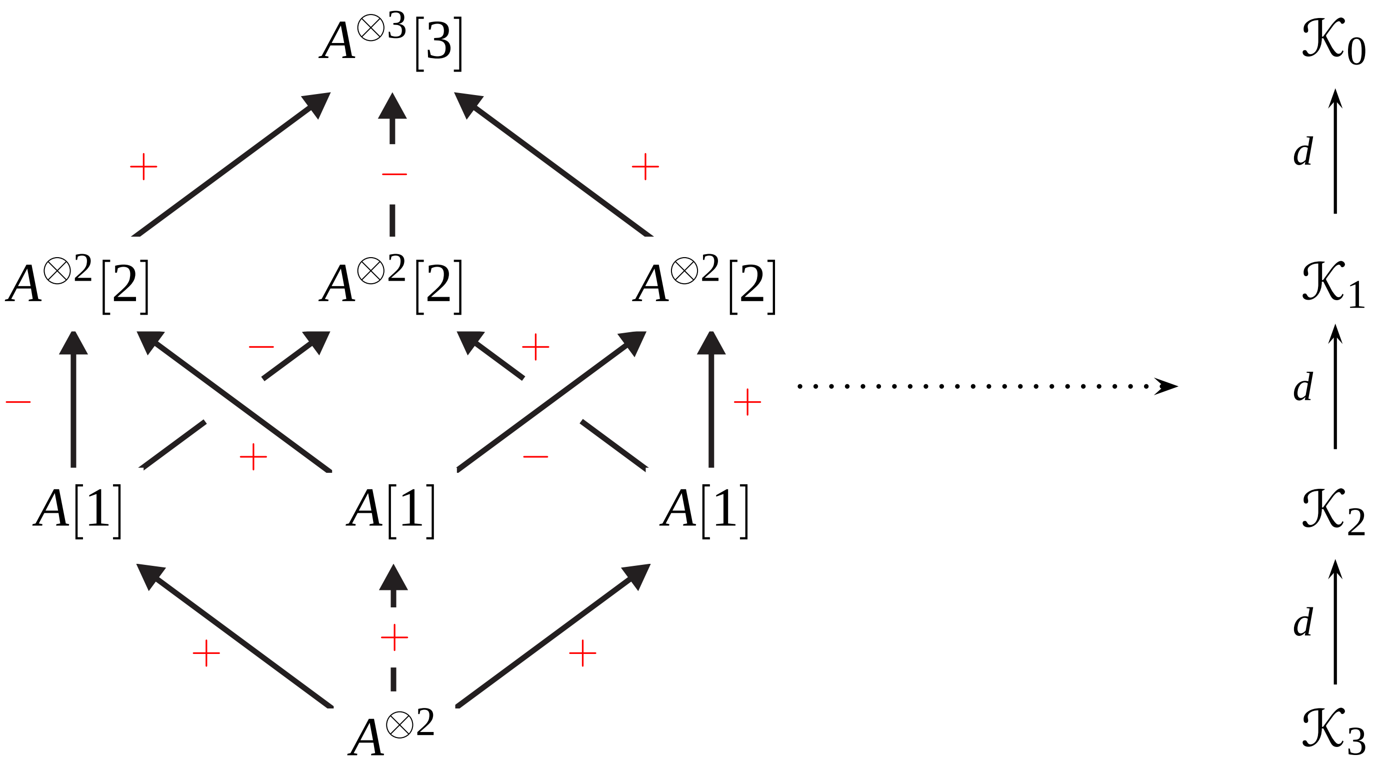
$m : A \otimes A \rightarrow A$



$\mu : A \rightarrow A \otimes A$



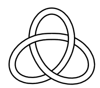


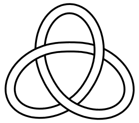


$\mathcal{K}_*$

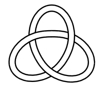
Khovanov (chain) complex

add $\pm$'s to edge maps so squares *anticommute*

Khovanov homology $KH_*\Big($  $, \mathbb{Q}\Big) = H_*(\mathcal{K}_*)$

	6	4	2	0	-2	$q$dim
KH_0	$\mathbb{Q}$					q^6
KH_1			$\mathbb{Q}$			q^2
KH_2						0
KH_3				$\mathbb{Q}$	$\mathbb{Q}$	$1+q^{-2}$

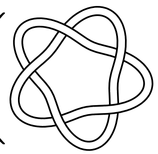
Euler characteristic $\chi(\mathcal{K}_*)$

$= \sum(-1)^i q\dim KH_i\Big($  $, \mathbb{Q}\Big)$

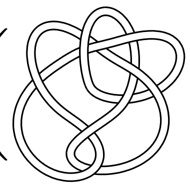
$$= q^6 - q^2 - 1 - q^{-2}$$

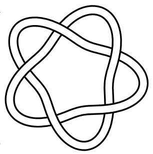
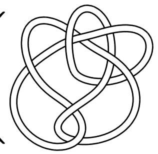
minor miracle: KH_* an invariant (after a bit of nudging)

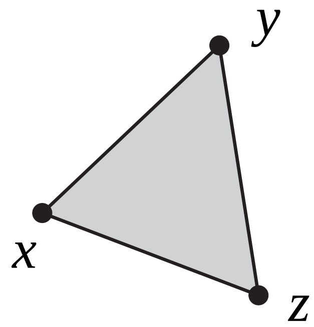
$\mathbb{Q}$						
		$\mathbb{Q}$				
		$\mathbb{Q}$				
				$\mathbb{Q}$		
					$\mathbb{Q}$	$\mathbb{Q}$

$KH_*\Big($  $\Big)$

$\mathbb{Q}$							
		$\mathbb{Q}$					
		$\mathbb{Q}$					
			$\mathbb{Q}$	$\mathbb{Q}$			
			$\mathbb{Q}$		$\mathbb{Q}$		
					$\mathbb{Q}\oplus\mathbb{Q}$		
							$\mathbb{Q}$
						$\mathbb{Q}$	$\mathbb{Q}$

$KH_*\Big($  $\Big)$

- Jones$\Big($  $\Big) =$ Jones$\Big($  $\Big)$
- FUNCTORIAL!!



iff $x < y < z$

- poset $P \longrightarrow |P|$ order (simplicial) complex.

- poset homology $=$ simplicial homology of $|P|$

ie: $H_*(P,R) := H_*(|P|,R) =$ homology of chain complex

$$C_n(P,R) = \bigoplus_{x_0 < \cdots < x_n} R$$

with differential $d : C_n(P,R) \to C_{n-1}(P,R)$

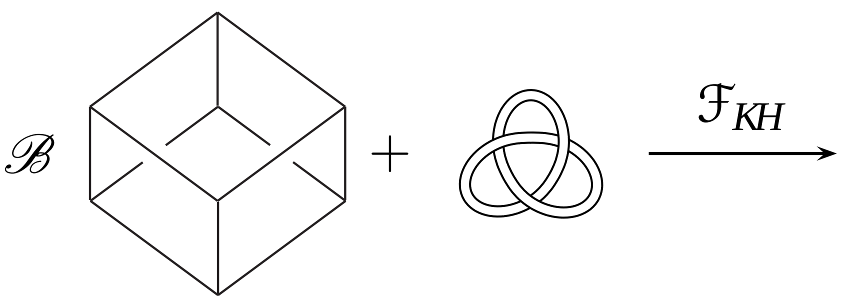
$$\lambda \cdot (x_0 < \cdots < x_n) \overset{d}{\mapsto} \sum_{j=0}^{n} (-1)^j \lambda \cdot (x_0 < \cdots < \widehat{x}_j < \cdots < x_n)$$

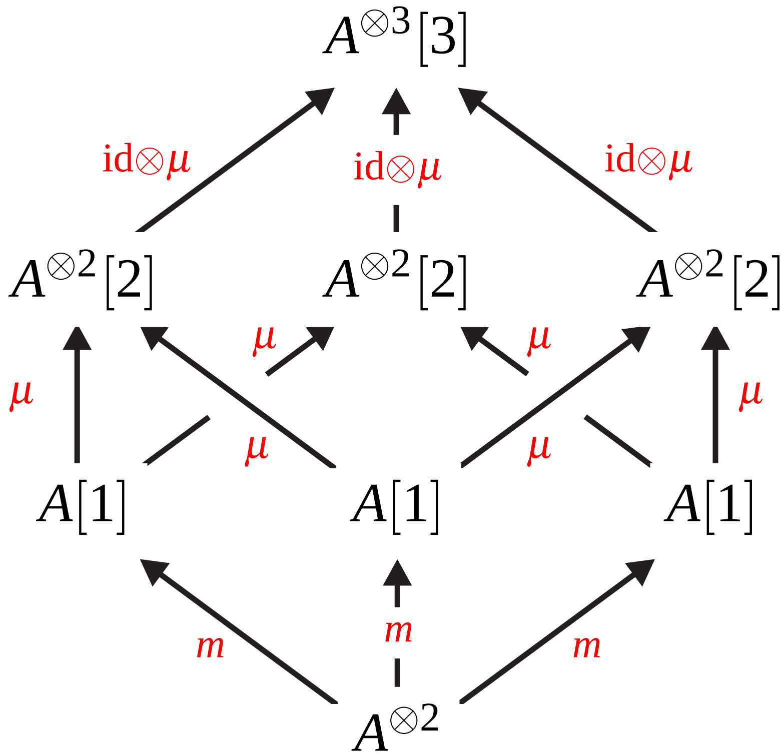
- Eg: [Folkman-Björner] P finite geometric lattice

$$\widetilde{H}_n(P \setminus \{0,1\}, \mathbb{Z}) = \begin{cases} \mathbb{Z}^{|\mu(0,1)|} & n = \text{rk}P - 2, \\ 0 & \text{otherwise.} \end{cases}$$

- $P \xrightarrow{\mathcal{F}} R$-**mod** (covariant) functor
(= pre-cosheaf of modules over P)

- Eg: "Khovanov colouring":





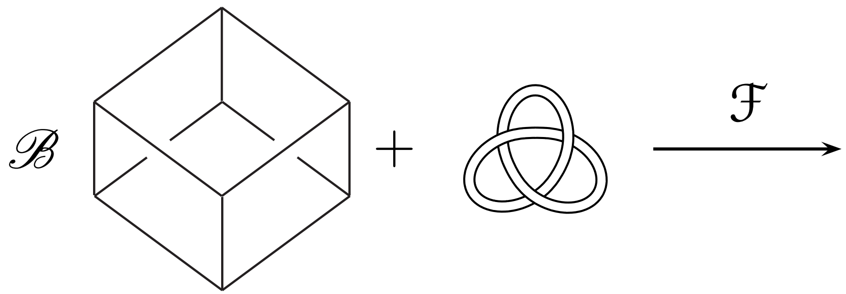
- $P \xrightarrow{\mathcal{F}} R$-**mod** sheaf

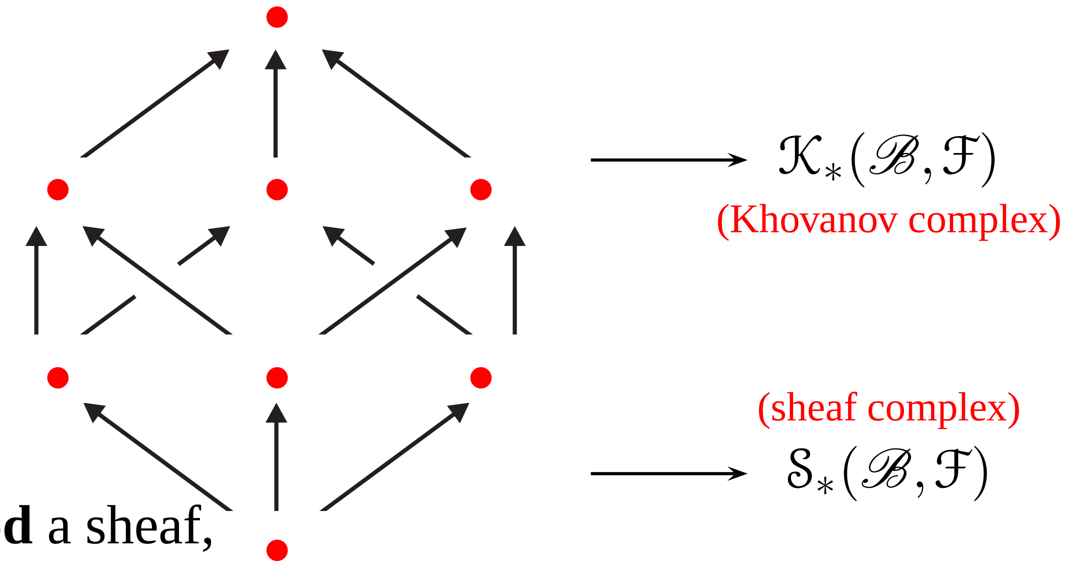
- sheaf homology $\mathcal{H}_*(P,\mathcal{F})$ = homology of chain complex

$$\mathcal{S}_n(P,\mathcal{F}) = \bigoplus_{x_0<\cdots<x_n} \mathcal{F}(x_0)$$

with differential $d : \mathcal{S}_n(P,\mathcal{F}) \to \mathcal{S}_{n-1}(P,\mathcal{F})$

$$\lambda \cdot (x_0 < \cdots < x_n) \overset{d}{\mapsto} \mathcal{F}(x_0 < x_1)(\lambda) \cdot (\widehat{x_0} < x_1 < \cdots < x_n) + \sum_{j=1}^{n} (-1)^j \lambda \cdot (x_0 < \cdots < \widehat{x_j} < \cdots < x_n)$$





- **Theorem**: $\mathscr{B}$ Boolean and $\mathscr{B} \xrightarrow{\mathcal{F}} R$-**mod** a sheaf,

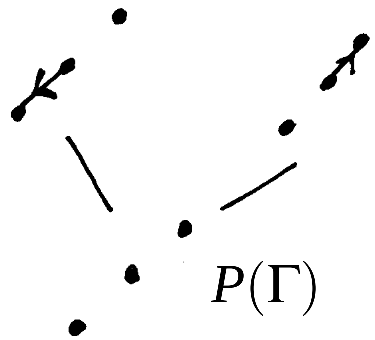
$$KH_*(\mathscr{B},\mathcal{F}) \cong \widetilde{\mathcal{H}}_{*-1}(\mathscr{B} \setminus 1,\mathcal{F})$$

- [generally: one can define a "cellular" homology $H_*^{\text{cell}}(P,\mathcal{F})$:

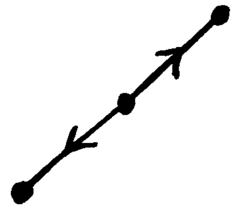
  Theorem: P "cellular" poset and $P \xrightarrow{\mathcal{F}} R$-**mod** a sheaf, then

$$H_*^{\text{cell}}(P,\mathcal{F}) \cong \mathcal{H}_*(P,\mathcal{F})$$

  Eg: $P =$ geometric lattices, cell posets regular CW-complexes, Cohen-Macaulay posets, ...]



$P(\Gamma)$



Γ

- $A =$ associative R-algebra.

- $P(\Gamma) =$ quiver poset of directed graph Γ.

- $P(\Gamma) \xrightarrow{\mathcal{F}_A}$ $A\otimes A \leftarrow A\otimes A\otimes A \rightarrow A\otimes A$

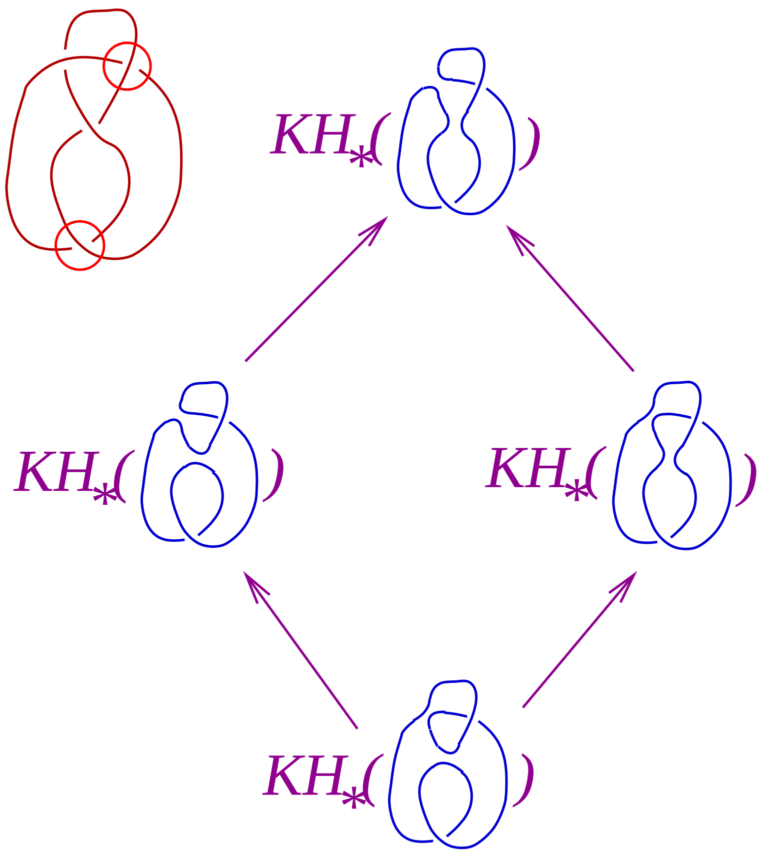
- "homology of A with coefficients in Γ"
  $:= \mathcal{H}_*(P(\Gamma), \mathcal{F}_A)$

- **Corollary** [Turner-Wagner]:

$$\mathcal{H}_i(P(n\text{-gon}), \mathcal{F}_A) \cong HH_i(A), \ (0 \leq i \leq n-1)$$

($HH_*(A) =$ Hochschild homology)

- Take an N-crossing link diagram D and fix k crossings.
- Resolve each of the remaining crossings as usual.
- Put the resulting 2^{N-k} diagrams on a Boolean lattice $\mathscr{B}$.
- Define a sheaf on $\mathscr{B}$ by taking $KH_*(-)$.



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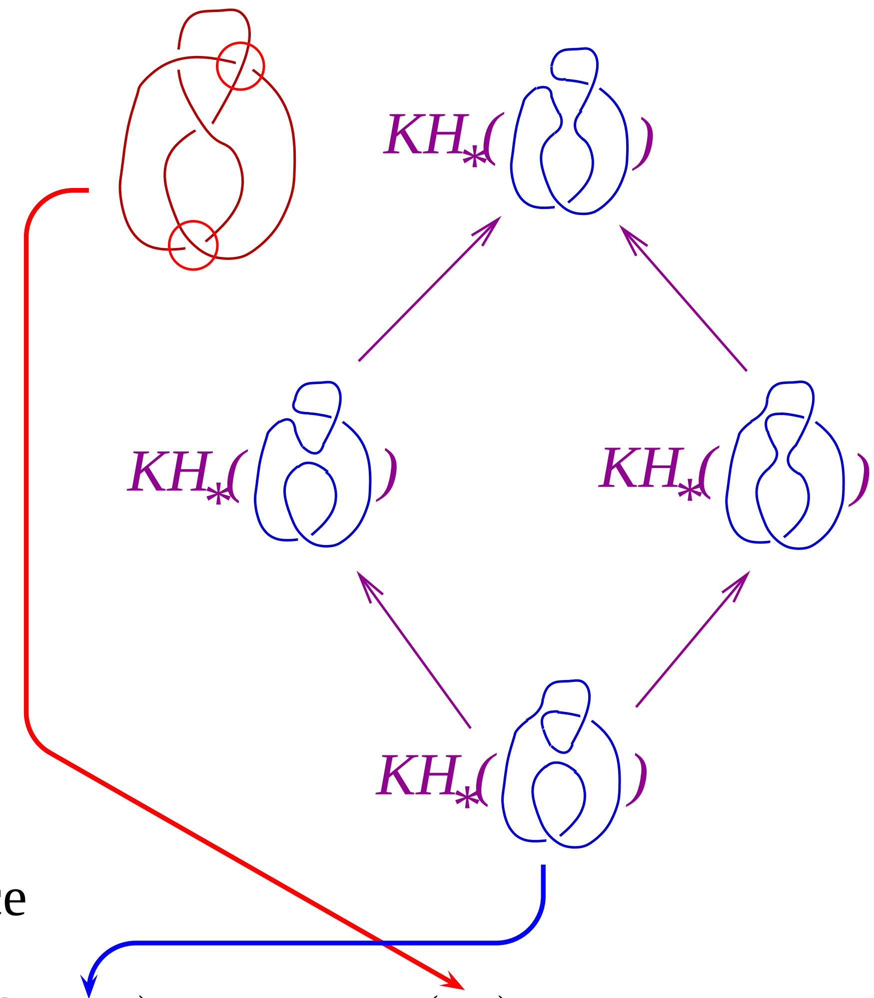

Theorem: There is a spectral sequence

$$E^2_{p,q} = KH_p(\mathscr{B}, KH_q) \Longrightarrow KH_{p+q}(\quad)$$