1 GSW… The Equivalent Baseband

With baseband modulation, there is only one property of the transmitted signal that can be modified to convey any information: the amplitude.

With passband modulation, there are two: the amplitude and the phase of the carrier. Any quantity with two independent components can be conveniently represented by a complex number, and this provides a very useful and powerful technique for analysing passband modulation formats. To use this technique, we'll need to develop the theory of baseband communications to cover complex signals. This can be a bit confusing at first, but the results at the end of the derivations are very simple, and very useful in analysing a wide range of modulation schemes.

1.1 Extending Communications Theory to Complex Signals

Imagine for a moment a complex world, where signals are complex, and filters have complex impulse responses. How does this affect the derivations of several key results in communications theory: in particular the derivation of the power spectral density of a modulated signal, and the optimum signal to noise ratio after a matched filter?

1.1.1 The Power Spectral Density of Complex Signals

First, consider the power spectral density of such a complex signal. You can take the Fourier transform of a complex signal with exactly the same formula as the Fourier transform of a real signal. The Fourier transform derives from the complex Fourier series, which can represent complex signals just as easily as real signals.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \exp(j\omega t)\, dt \quad (0.1)$$

The only difference is that with real signals, $X(\omega)$ is symmetric in the sense that:

$$X(\omega) = X^*(-\omega) \quad (0.2)$$

whereas with complex signals, $X(\omega)$ doesn't necessarily have any such symmetry.

The energy spectral density of a signal is the square of the amplitude of the Fourier transform of the signal:

$$ESD(\omega) = |X(\omega)|^2 \quad (0.3)$$

and for a continuous signal with infinite energy, the power spectral density is the mean value of the energy spectral density of one-second samples of the signal with random data:

$$PSD(\omega) = \overline{|Y(\omega)|^2} \quad (0.4)$$

where $Y(\omega)$ is the Fourier transform of a one-second long sample of a continuous modulated signal $y(t)$.

All that is exactly the same as for real signals (with the exception of the symmetry of the Fourier transform).

1.1.2 Matched Filtering of Complex Signals

The theory is derived in the same way as for a real signal: start with an impulse that is fed through a transmit filter with an impulse response $h(t)$. The difference now is that $h(t)$ can be complex, so the transmitted waveform has both a real and imaginary component. In this case, the signal emerging from the transmitter is the result of putting an impulse at time zero into a filter with an impulse response of $h(t)$, in other words it's just $h(t)$.

Receive this signal using a receive filter with an impulse response $g(t)$, and the received signal will be:

$$r(t) = \int_{-\infty}^{\infty} g(\tau)h(t-\tau)\,d\tau \tag{0.5}$$

If the noise is white, with a one-sided noise density of N_0 W/Hz, then the two-sided noise density (considering both positive and negative frequencies) will be $N_0/2$ W/Hz or $N_0/4\pi$ W/rad/s which makes the received noise power at the output of the filter equal to:

$$N = \frac{N_0}{4\pi}\int_{-\infty}^{\infty} \left|G(\omega)\right|^2 d\omega \tag{0.6}$$

and applying Parseval's theorem[1] to this gives:

$$N = \frac{N_0}{2}\int_{-\infty}^{\infty} \left|g(t)\right|^2 dt \tag{0.7}$$

Therefore the maximum possible signal to noise ratio at the output of the receive filter at time t is:

$$SNR = \frac{\left(\int_{-\infty}^{\infty} g(\tau)h(t-\tau)\,d\tau\right)^2}{N = \frac{N_0}{2}\int_{-\infty}^{\infty} \left|g(t)\right|^2 dt} \tag{0.8}$$

[1] Parseval's theorem states that the total energy in a transmitted symbol can be calculated from either integrating the power over all time, or integrating the energy spectral density over all frequencies: the result should be the same. With complex signals, the power in the signal at any time is the square of the modulus of the complex signal $x(t)$, therefore:

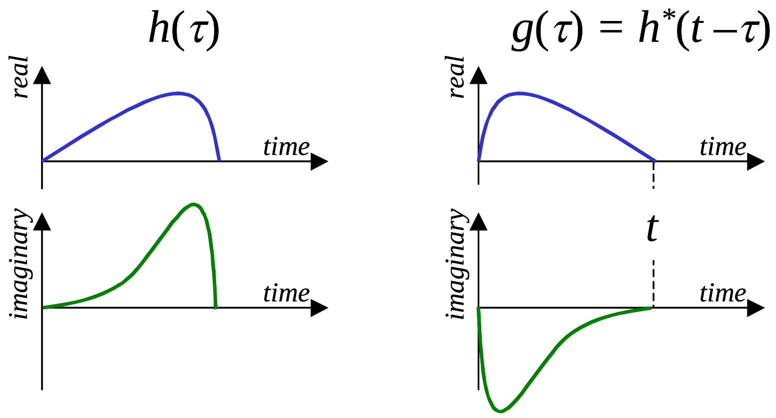
$$\int_{-\infty}^{\infty} \left|h^2(t)\right| dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|H(\omega)\right|^2 d\omega$$

For the reason why the factor of 2π is there, see the chapter on Fourier analysis.

and this can be maximised by ensuring that:

$$g(\tau) = h^*(t-\tau) \tag{0.9}$$

which is similar to the result derived for real signals, the only difference being that the optimum receiver filter now has the complex conjugate of the time-reversed transmit filter's impulse response. For example:



In this case, the maximum signal to noise ratio can be calculated as:

$$\begin{aligned} SNR &= \frac{\left(\int_{-\infty}^{\infty} g(\tau) g^*(\tau) d\tau\right)\left(\int_{-\infty}^{\infty} h^*(t-\tau) h(t-\tau) d\tau\right)}{\frac{N_0}{2}\int_{-\infty}^{\infty} |g(t)|^2 dt} \\ &= \frac{2\left(\int_{-\infty}^{\infty} |g(\tau)|^2 d\tau\right)\left(\int_{-\infty}^{\infty} |h(t-\tau)|^2 d\tau\right)}{N_0 \int_{-\infty}^{\infty} |g(t)|^2 dt} \\ &= \frac{2\int_{-\infty}^{\infty} |h(t-\tau)|^2 d\tau}{N_0} \end{aligned} \tag{0.10}$$

The numerator is the total energy in a time-reversed version of the transmitted waveform, which must be the same as the energy in the transmitted waveform, so we can write:

$$SNR = \frac{2E_s}{N_0} \tag{0.11}$$

where E_s is the energy in the symbol (the integral of the magnitude of the signal over all time). This is exactly the same result as we get for real signals.

1.2 A Complex Representation of Amplitude and Phase

Any passband signal (for that matter any signal at all), can be expressed in terms of a carrier frequency:

$$x(t) = A(t)\cos\left(\omega_c t + \phi(t)\right) \tag{0.12}$$

where $A(t)$ is the amplitude of the signal, and $\phi(t)$ is the relative phase of the signal relative to the phase of the carrier, both being functions of time. After all, the only things that can distinguish the pure unmodulated carrier frequency and the modulated signal are the modulated signal's amplitude and its phase[2].

This signal $x(t)$ can also be written as the real part of a complex variable, for which we are free to define the imaginary part any way we like. However, there is a standard complex exponential way of representing frequencies which allows a particularly simple and convenient notation, and this suggests using:

$$x(t) = \Re\left\{A(t)\cos\left(\omega_c t + \phi(t)\right) + j\,A(t)\sin\left(\omega_c t + \phi(t)\right)\right\} \tag{0.13}$$

Euler's equation[3] applied to the above equation gives:

$$x(t) = \Re\left\{A(t)\exp\left(j\left(\omega_c t + \phi(t)\right)\right)\right\} \tag{0.14}$$

let $x_{\mathbf{E}}(t) = A(t)\exp(j\,\phi(t))$, and we get:

$$x(t) = \Re\left\{x_{\mathbf{E}}(t)\exp\left(j\omega_c t\right)\right\} \tag{0.15}$$

$x_{\mathbf{E}}(t)$ is known as the equivalent baseband representation of the signal. It's really just the difference between the signal and the unmodulated carrier frequency, expressed in complex terms. The amplitude of $x_{\mathbf{E}}(t)$ is the amplitude of the transmitted signal $A(t)$, and the phase of $x_{\mathbf{E}}(t)$ is the phase difference between the transmitted signal and the carrier $\phi(t)$.

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[2] You'll notice that there isn't a unique choice of $A(t)$ and $\phi(t)$ to represent any signal, for example a simple cosine wave at a frequency of ω_c can be represented either by $A(t) = 1$ and $\phi(t) = 0$, or by $A(t) = \cos(\omega_c t)$, $\phi(t) = -\omega_c t$, or any of an infinite number of other possibilities. The usual choice of ω_c, $A(t)$ and $\phi(t)$ are those which result in the most slowly changing $A(t)$ and $\phi(t)$. Usually the correct choice of carrier frequency, $A(t)$ and $\phi(t)$ is obvious.

[3] See the chapter on complex numbers if unfamiliar with Euler's equation: $\exp(j\theta) = \cos(\theta) + j\sin(\theta)$.

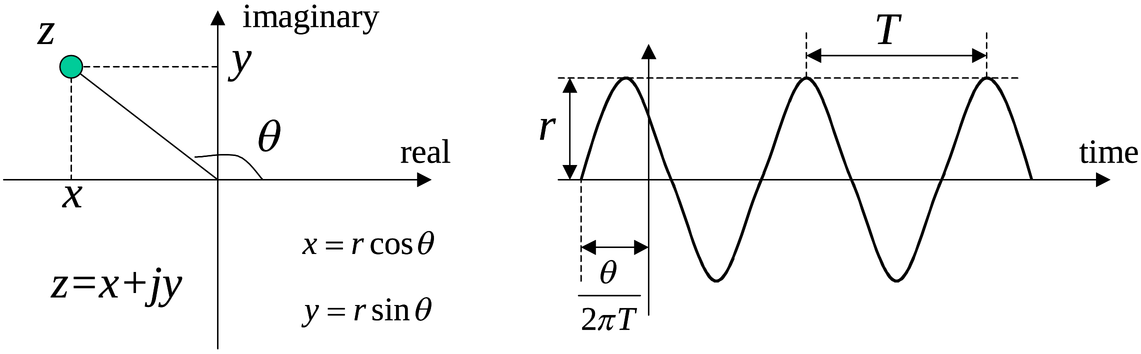

Figure 1-1 – Illustration of the Complex Baseband Representation

The real part of the equivalent baseband representation, shown as *x* in Figure 1-1 is also known as the in-phase or *I* component, and the value of *y* as the quadrature or *Q* component.

Some more examples:

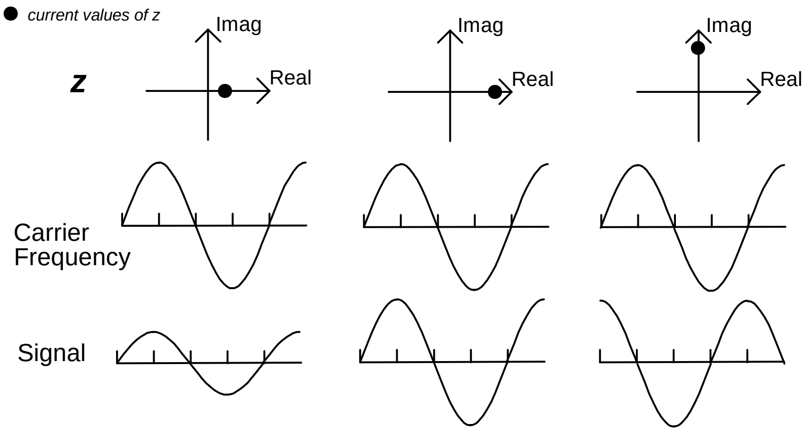

Figure 1-2 –Examples of Complex Baseband Representation

1.2.1 A Note About Notation

Dealing with the equivalent baseband can get a bit confusing, since we'll be dealing with real baseband signals, complex baseband signals and real passband signals. We'll need an obvious way to tell them apart. Here, any real signal (whether baseband or passband) will be written as a normal variable, for example:

$$x(t),\ X(\omega),\ N_0,\ E_s$$

are a signal, the Fourier transform of that signal, the one-sided Noise Spectral Density and the energy in a symbol respectively. All simple, real quantities. They might be baseband or passband signals, but they will always be real.

I'll write the equivalent baseband representation of a passband quantity with a bold, non-italic capital E suffix, so that, for example:

$$x_{\mathbf{E}}(t),\ X_{\mathbf{E}}(\omega),\ N_{0\mathbf{E}},\ E_{s\mathbf{E}}$$

are the equivalent baseband representation of a passband signal, the Fourier transform of the equivalent baseband signal, the equivalent baseband one-sided Noise spectral density, and the energy in an equivalent baseband signal respectively. (This is not a standard notation, but there doesn't seem to be a good standard notation that makes the difference between the equivalent baseband signals and the passband signals obvious.)

1.2.2 Signal Spectra in the Equivalent Baseband Representation

Suppose we have a passband signal $x(t)$ with a spectrum of $X(\omega)$. What is the spectrum of the equivalent baseband representation $x_{\mathbf{E}}(t)$?

Noting the very useful identity:

$$\Re\{z\} = \frac{z + z^*}{2} \tag{0.16}$$

where $\Re\{z\}$ is the real part of the complex number z, allows the real signal $x(t)$ to be represented as:

$$x(t) = \Re\{x_{\mathbf{E}}(t)\exp(j\omega_c t)\} = \frac{x_{\mathbf{E}}(t)\exp(j\omega_c t) + x_{\mathbf{E}}^*(t)\exp(-j\omega_c t)}{2} \tag{0.17}$$

Taking the Fourier transform of this expression gives:

$$\begin{aligned} X(\omega) &= \frac{1}{2}\int_{-\infty}^{\infty} x_{\mathbf{E}}(t)\exp(j\omega_c t)\exp(-j\omega t)\, dt + \frac{1}{2}\int_{-\infty}^{\infty} x_{\mathbf{E}}^*(t)\exp(-j\omega_c t)\exp(-j\omega t)\, dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} x_{\mathbf{E}}(t)\exp(-j(\omega - \omega_c)t)\, dt + \frac{1}{2}\int_{-\infty}^{\infty} x_{\mathbf{E}}^*(t)\exp(j(-\omega - \omega_c)t)\, dt \\ &= \frac{1}{2}X_{\mathbf{E}}(\omega - \omega_c) + \frac{1}{2}X_{\mathbf{E}}^*(-(\omega + \omega_c)) \end{aligned} \tag{0.18}$$

where $X_{\mathbf{E}}(\omega)$ is the Fourier transform of the equivalent baseband signal, defined as:

$$X_{\mathbf{E}}(\omega) = \int_{-\infty}^{\infty} x_{\mathbf{E}}(t)\exp(-j\omega t)\, dt \tag{0.19}$$

$X_{\mathbf{E}}(\omega - \omega_c)$ is simply a copy of the Fourier transform of the equivalent baseband signal shifted up in frequency by the carrier frequency ω_c. $X_{\mathbf{E}}^*(-\omega - \omega_c)$ is the complex conjugate of the Fourier transform of the equivalent baseband signal reversed in time, and shifted down by the carrier frequency.

This result means that the Fourier transform of the passband signal $X(\omega)$ can be derived from the Fourier transform of the equivalent baseband signal $X_{\mathbf{E}}(\omega)$ by dividing $X_{\mathbf{E}}(\omega)$ by two and shifting up in frequency by the carrier frequency to form $0.5X_{\mathbf{E}}(\omega - \omega_c)$, and then adding the complex conjugate of the reflection of this spectrum in the frequency axis (as we have to do to make sure that the passband signal is real).

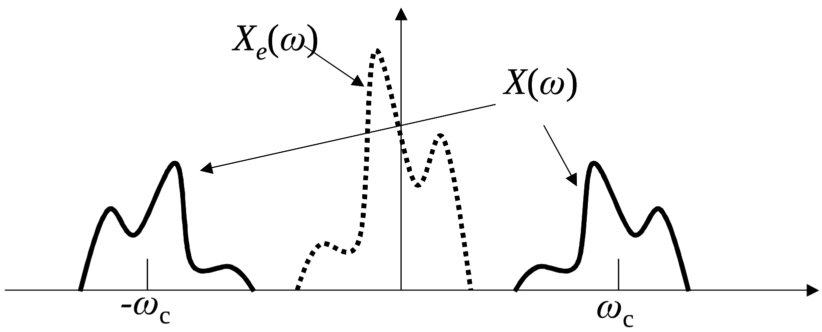

Figure 1-3 –Frequency Spectrum of the Equivalent Baseband Representation

Looking at this result from another angle, suppose we wanted to work out $X_{\mathbf{E}}(\omega)$ from $X(\omega)$, the Fourier transform of the equivalent baseband signal can be obtained from the Fourier transform of the real passband signal by deleting all the negative frequency components[4], moving the positive frequency components down by the carrier frequency, and then multiplying by two[5].

1.2.3 A Simple Example of an Equivalent Baseband Spectrum

A very simple example might help to illustrate this point. Suppose we have an equivalent baseband signal that is a single complex frequency with an amplitude of *A*:

$$x_{\mathbf{E}}(t) = A\exp(j\upsilon t) \tag{0.20}$$

This has a Fourier transform of:

$$X_{\mathbf{E}}(\omega) = \int_{-\infty}^{\infty} A\exp\left(j(\upsilon-\omega)t\right) dt = A\delta(\upsilon-\omega) \tag{0.21}$$

In the real passband, this represents a signal of:

$$\begin{aligned} x(t) &= \Re\left\{A\exp\left(j(\upsilon+\omega_c)t\right)\right\} \\ &= A\cos\left((\upsilon+\omega_c)t\right) \end{aligned} \tag{0.22}$$

and the Fourier transform of a cosine is:

[4] We're not losing any information here, since the negative frequency components have to be the complex conjugates of the positive frequency components to ensure the passband signal is real.

[5] Note this only works if the carrier frequency is greater than the bandwidth of the signal, however this is true for all systems I'll be discussing here.

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} A\cos\left(\left(\upsilon+\omega_c\right)t\right)\exp\left(-j\omega t\right)dt \\
&= \frac{A}{2}\int_{-\infty}^{\infty}\exp\left(j\left(\upsilon+\omega_c-\omega\right)t\right)dt+\frac{A}{2}\int_{-\infty}^{\infty}\exp\left(j\left(-\upsilon-\omega_c-\omega\right)t\right)dt \\
&= \frac{A}{2}\delta\left(\upsilon+\omega_c-\omega\right)+\frac{A}{2}\delta\left(-\upsilon-\omega_c-\omega\right)
\end{aligned}
\tag{0.23}
$$

the result has two peaks, each exactly half the amplitude of the peak in the Fourier transform of the equivalent baseband representation of the signal.

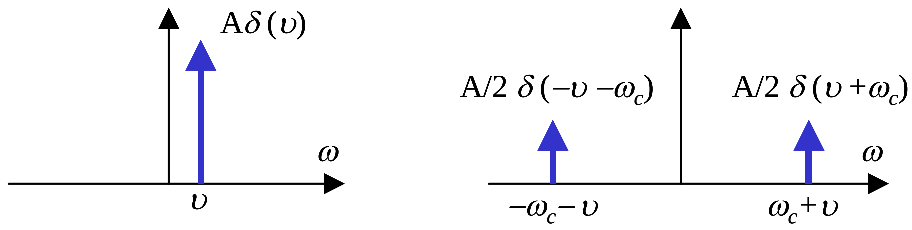

Figure 1-4 A Simple Example of Equivalent Baseband Spectrum

1.3 The I-Q Modulator and the Equivalent Baseband

Since there are two parameters to modify when creating a passband signal (the amplitude and phase of the carrier), there must be two inputs to the modulator. One is called the in-phase component *I*(*t*), and the other the quadrature component *Q*(*t*). A typical modulator might then look like this:

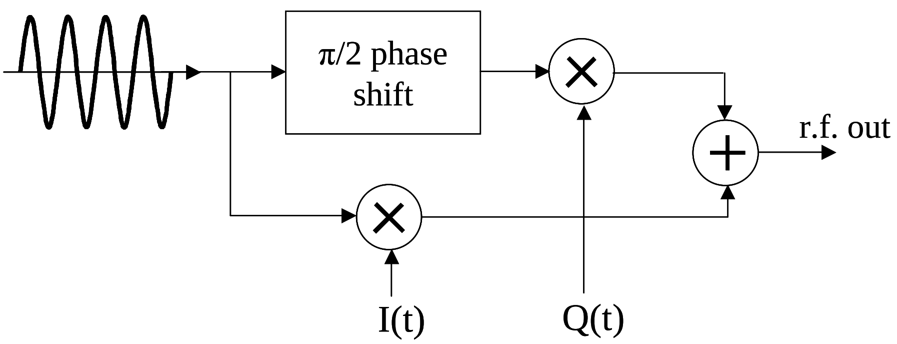

Figure 1-5 – A Schematic Modulator

and will produce a signal according to:

$$
\begin{aligned}
x(t) &= I(t)\cos(\omega_c t)+Q(t)\cos\left(\omega_c t+\frac{\pi}{2}\right) \\
&= I(t)\cos(\omega_c t)-Q(t)\sin(\omega_c t)
\end{aligned}
\tag{0.24}
$$

Consider the real passband signal represented by an equivalent baseband signal of:

$$
x_{\mathrm{E}}(t)=\left(I(t)+jQ(t)\right) \tag{0.25}
$$

this produces:

$$\begin{aligned} x(t) &= \Re\left\{ x_{\mathbf{E}}(t)\exp(j\omega_c t) \right\} \\ &= \Re\left\{ \left(I(t) + jQ(t) \right)\left(\cos(\omega_c t) + j\sin(\omega_c t) \right) \right\} \\ &= I(t)\cos(\omega_c t) - Q(t)\sin(\omega_c t) \end{aligned} \tag{0.26}$$

which is exactly the signal produced by the modulator. In other words, the equivalent baseband representation of the passband signal produced by this modulator has a real part equal to the value of $I(t)$, and an imaginary component equal to the value of $Q(t)$. This is a common technique for generating passband signals.

Receiving passband signals can be done in a very similar way, using a copy of the carrier frequency and a version of the carrier frequency 90 degrees ($\pi/2$ radians) out of phase:

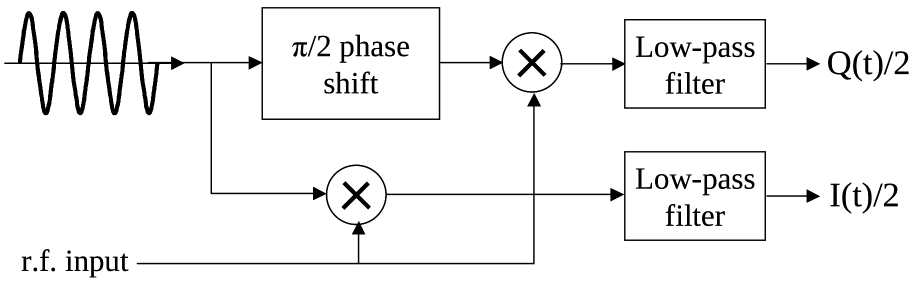

Figure 1-6 – A Schematic Demodulator

In maths:

$$\begin{aligned} \left\{ I(t)\cos(\omega_c t) - Q(t)\sin(\omega_c t) \right\}\cos(\omega_c t) &= I(t)\cos^2(\omega_c t) - Q(t)\cos(\omega_c t)\sin(\omega_c t) \\ &= I(t)\left(\frac{1+\cos(2\omega_c t)}{2} \right) - Q(t)\left(\frac{\sin(2\omega_c t)}{2} \right) \end{aligned} \tag{0.27}$$

and using the low-pass filter to get rid of the components at twice the carrier frequency gives $I(t)/2$.

Similarly,

$$\begin{aligned} \left\{ I(t)\cos(\omega_c t) - Q(t)\sin(\omega_c t) \right\}\sin(\omega_c t) &= I(t)\sin(\omega_c t)\cos(\omega_c t) - Q(t)\sin^2(\omega_c t) \\ &= I(t)\left(\frac{\sin(2\omega_c t)}{2} \right) - Q(t)\left(\frac{1-\cos(2\omega_c t)}{2} \right) \end{aligned} \tag{0.28}$$

and again filtering out the high-frequency components provides an easy way to recover $Q(t)$.

1.4 Noise in the Equivalent Baseband – Part One

Just as with baseband communication systems, we usually quote the noise power in terms of the real, passband one-sided noise spectral density N_0, expressed in W/Hz. The one-sided noise spectral density does not include negative frequencies, and is exactly twice the two-sided noise spectral density, which does includes negative frequencies[6].

Before we can start applying baseband communication theory to the equivalent baseband, we need to know how much noise there is in the equivalent baseband. There are several ways to derive the result, but one simple method goes as follows: first, consider a receiver that samples the incoming noise through a bandpass filter that removes all of the noise except the noise in a bandwidth B Hz (that's $2\pi B$ rad/s) around the carrier frequency, where the signal lies. The noise power spectral density at the output of this filter will therefore look like this:

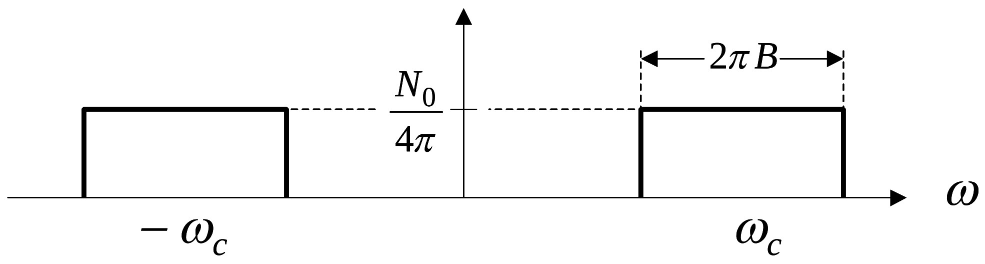

Figure 1-7 – Noise Power Spectral Density in a Bandwidth *B* around the Carrier Frequency

(Here, the one-sided noise spectral density in terms of Watts per Hertz is N_0, so the one-sided noise spectral density in terms of Watts per rad/s is $N_0/2\pi$, and the two-sided noise spectral density as shown is $N_0/4\pi$ W/rad/s).

We know that the Fourier transform of a signal in the equivalent baseband $X_e(\omega)$ can be derived from the Fourier transform of the signal in the real passband $X(\omega)$ by deleting all the negative frequency components, moving the positive frequency components down in frequency by the carrier frequency, and multiplying by two.

What about the power spectral density? The power is proportional to the square of the amplitude, so the power spectral density is proportional to the square of the modulus of the Fourier transform. And if the Fourier transform in the equivalent baseband has twice the magnitude of the Fourier transform of the passband signal, then the power spectral density in the equivalent baseband must be four times the power spectral density in the passband.

So, if the noise spectral density is $N_0/4\pi$ W/rad/s in the passband, it must be N_0/π W/rad/s in the equivalent baseband.

[6] If this is unfamiliar, look in the chapter on noise.

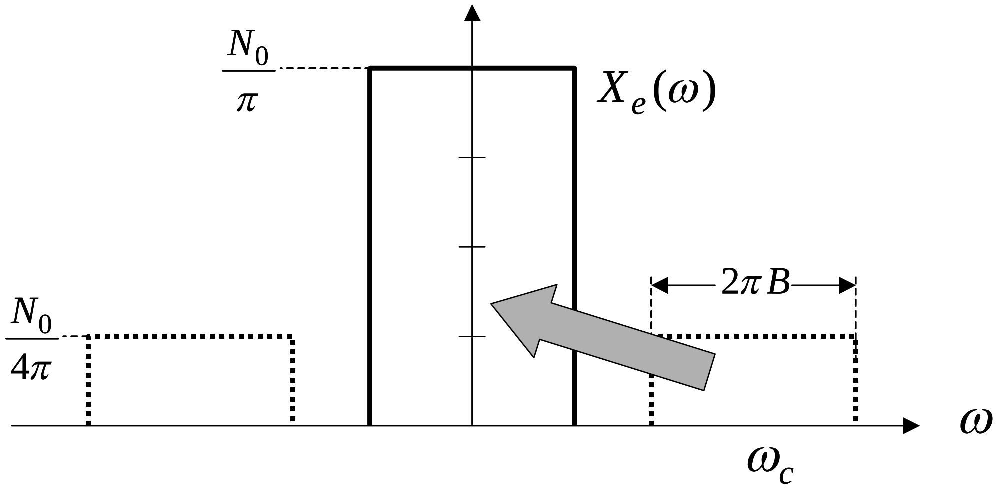

Figure 1-8 – Converting Noise Spectral Density to the Equivalent Baseband

Or terms of W/Hz, a one-sided noise power spectral density of N_0 W/Hz in the real passband corresponds to a two-sided noise spectral density of $2N_0$ W/Hz in the equivalent baseband; which is a one-sided equivalent baseband noise spectral density of $4N_0$.

1.4.1 Another Way of Deriving the Equivalent Baseband Noise

For those more mathematically inclined: consider a long sample (T seconds long) of a real passband noise signal $n(t)$ in equivalent baseband form $n_{\mathbf{E}}(t)$, so that:

$$n(t) = \Re\left\{n_{\mathbf{E}}(t)\exp(j\omega_c t)\right\} \tag{0.29}$$

In the passband, assume that this noise results from a noise spectral density N_0 acting in a bandwidth B, so that the variance (the mean square) value of the passband noise $n(t)$ is N_0B. What is the mean square of the magnitude of the equivalent baseband noise $n_{\mathbf{E}}(t)$? This should be equal to $N_{0\mathbf{E}}B/2$, since a bandwidth of B in the passband corresponds to a one-sided bandwidth of $B/2$ in the equivalent baseband. In other words:

$$\left|n_{\mathbf{E}}(t)\right|^2 = n_{\mathbf{E}}(t)n_{\mathbf{E}}^*(t) = N_{0\mathbf{E}}\frac{B}{2} \tag{0.30}$$

We can expand equation (0.29) to get:

$$n(t) = \frac{n_{\mathbf{E}}(t)\exp(j\omega_c t) + n_{\mathbf{E}}^*(t)\exp(-j\omega_c t)}{2} \tag{0.31}$$

and taking the mean (expectation) value of the square of this gives:

$$\begin{aligned}\overline{n^2(t)} &= \frac{1}{4}\left(n_{\mathbf{E}}(t)\exp(j\omega_c t) + n_{\mathbf{E}}^*(t)\exp(-j\omega_c t)\right)^2 \\ &= \frac{1}{4}\left(\begin{array}{l}\overline{n_{\mathbf{E}}^2(t)\exp(2j\omega_c t)} + \overline{\left(n_{\mathbf{E}}^*(t)\right)^2\exp(-2j\omega_c t)} \\ +2\overline{n_{\mathbf{E}}(t)n_{\mathbf{E}}^*(t)}\end{array}\right)\end{aligned} \tag{0.32}$$

Now the first two terms on the right-hand side are very fast oscillations (at twice the carrier frequency), so provided the carrier frequency is much greater that the rate at which the equivalent baseband noise is changing (which is usually the case, since the only interesting equivalent baseband noise is around the frequencies of the equivalent baseband signal, and that's usually much slower than the carrier frequency), we can assume these will average out to zero. For example:

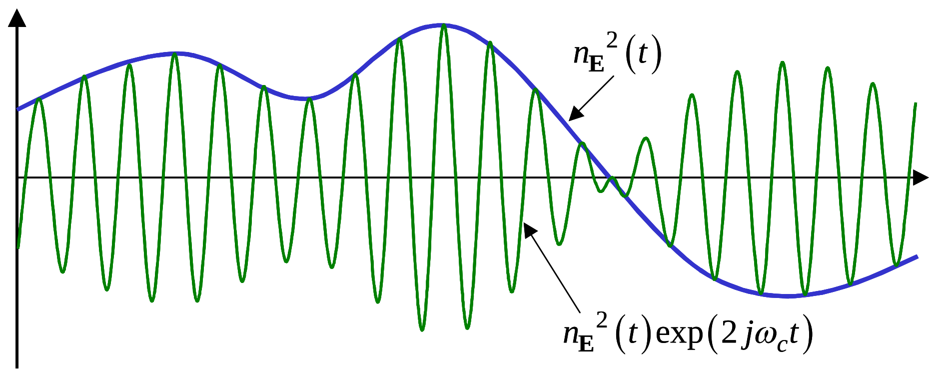

Figure 1-9 Averaging Out the Fast-Moving Terms

the product of the slow equivalent-baseband noise signal and the fast moving carrier will average out to zero over very short time-periods.

That leaves:

$$\overline{n^2(t)} = \frac{1}{2}\overline{n_{\mathbf{E}}(t)\,n_{\mathbf{E}}^{*}(t)} = \frac{1}{2}\overline{\left|n_{\mathbf{E}}(t)\right|^2} \tag{0.33}$$

Now we know from above that:

$$\overline{n^2(t)} = N_0 B \tag{0.34}$$

and from equation (0.30) that:

$$\overline{\left|n_{\mathbf{E}}(t)\right|^2} = N_{0\mathbf{E}}\frac{B}{2} \tag{0.35}$$

and therefore:

$$N_0 = \frac{N_{0\mathbf{E}}}{4} \tag{0.36}$$

The equivalent baseband one-sided noise spectral density is four times the real passband one-sided noise spectral density.

1.5 Optimum Filtering for Passband Communications

We saw back in section 1.1.2 that the well-known result for real signals:

$$SNR_{opt} = \frac{2E_s}{N_0} \tag{0.37}$$

is also true for complex signals, such as the equivalent baseband representation of passband signals. We could, for example, write:

$$S_{\mathbf{E}} N_{\mathbf{E}} R_{opt} = \frac{2E_{s\mathbf{E}}}{N_{0\mathrm{E}}} \tag{0.38}$$

where $S_{\mathbf{E}}$ is the instantaneous equivalent baseband signal power at the optimum sampling time, $N_{\mathbf{E}}$ is the equivalent baseband noise power, $E_{s\mathbf{E}}$ is the energy in one equivalent baseband symbol, and $N_{0\mathbf{E}}$ is the one-sided equivalent baseband noise spectral density.

It turns out to be very useful to express the equivalent baseband optimal signal to noise ratio in terms of the real passband symbol energy, and the real passband one-sided noise spectral density. We already know that:

$$N_{0\mathbf{E}} = 4N_0 \tag{0.39}$$

but what about $E_{S\mathbf{E}}$?

From the definition of E_{sE}, we know that the energy in the equivalent baseband symbol is the integral of the equivalent baseband power (i.e. the square of the magnitude of the equivalent baseband representation) over time:

$$E_{s\mathbf{E}} = \int_{-\infty}^{\infty} \left| x_{\mathbf{E}}(t) \right|^2 dt \tag{0.40}$$

Using a similar technique as before, we can note that for this symbol:

$$\begin{aligned} x(t) &= \Re\left\{ x_{\mathbf{E}}(t) \exp\left(j\omega_c t \right) \right\} \\ &= \frac{x_{\mathbf{E}}(t) \exp\left(j\omega_c t \right) + x_{\mathbf{E}}^*(t) \exp\left(-j\omega_c t \right)}{2} \end{aligned} \tag{0.41}$$

and squaring this gives:

$$x^2(t) = \frac{1}{4} \begin{pmatrix} x_{\mathbf{E}}^2(t) \exp\left(2j\omega_c t \right) + \left(x_{\mathbf{E}}^*(t) \right)^2 \exp\left(-2j\omega_c t \right) \\ +2x_{\mathbf{E}}(t) x_{\mathbf{E}}^*(t) \end{pmatrix} \tag{0.42}$$

The first two terms are the product of very fast oscillations (at twice the carrier frequency) with the much more slowly changing equivalent baseband representations of the signals. Therefore these terms will integrate out to zero. That leaves:

$$\int_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2} \int_{-\infty}^{\infty} \left| x_{\mathbf{E}}(t) \right|^2 dt \tag{0.43}$$

so:

$$E_s = \frac{1}{2} E_{s\mathbf{E}} \tag{0.44}$$

The equivalent baseband symbol energy is twice the real passband symbol energy.

There's an intuitive way of thinking about this result: an equivalent baseband symbol with a magnitude of A has an equivalent baseband power of A^2. However, it's really representing a cosine wave with a magnitude of A, and a cosine wave of magnitude A has a power of $A^2/2$, since:

$$\begin{aligned}\text{Power} &= \frac{1}{2\pi}\int_0^{2\pi} A^2\cos^2\left(\omega_c t\right)\,dt \\ &= \frac{A^2}{2\pi}\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{2}\cos\left(2\omega_c t\right)\right)dt \\ &= \frac{A^2}{4\pi}\int_0^{2\pi} dt + \frac{A^2}{4\pi}\int_0^{2\pi}\cos\left(2\omega_c t\right)dt \\ &= \frac{A^2}{2}\end{aligned} \tag{0.45}$$

so if the power at any moment in the equivalent baseband is twice the power in the real passband, the symbol energy in the equivalent baseband must be twice the energy in the real passband too.

So $N_{0\mathbf{E}} = 4N_0$ and $E_{s\mathbf{E}} = 2E_s$, and:

$$S_{\mathbf{E}}N_{\mathbf{E}}R_{opt} = \frac{2E_{s\mathbf{E}}}{N_{0\mathbf{E}}} = \frac{4E_s}{4N_0} = \frac{E_s}{N_0} \tag{0.46}$$

All the same conditions apply for this result to be valid as for the corresponding result for real baseband signals:

- S_e is the instantaneous received signal power at the optimum sampling time;
- The receive filter is the optimum (matched) filter;
- The delay through the receive filter is sufficient to allow all the energy in the received pulse to arrive[7];
- The noise is white;
- There is no intersymbol interference.

1.6 Noise in the Equivalent Baseband – Part Two

If $n_{\mathbf{E}}(t)$ is a complex noise signal in the equivalent baseband (the one with a one-sided noise spectral density of $4N_0$), then the component of noise in the $I(t)$ signal at the receiver is the real component of $n_{\mathbf{E}}(t)$, and the noise in the $Q(t)$ signal at the receiver is the imaginary component of $n_{\mathbf{E}}(t)$.

[7] In theory this can never happen in practice, although it's often so close to true that this assumption makes no practical difference.

We could write:

$$n_{\mathbf{E}}(t) = n_I(t) + jn_Q(t) \tag{0.47}$$

where $n_I(t)$ is the noise in the in-phase received signal, and $n_Q(t)$ is the noise in the quadrature received signal: both are real baseband quantities. These two noise components are independent: the value of $n_I(t)$ does not in any way depend on the current or past values of $n_Q(t)$.

Therefore, the total mean noise power in the equivalent baseband is:

$$\begin{aligned}\overline{\left|n_{\mathbf{E}}(t)\right|^2} &= \overline{\left|n_I(t) + jn_Q(t)\right|^2} = \overline{\left|n_I(t) + jn_Q(t)\right|\left|n_I(t) - jn_Q(t)\right|} \\ &= \overline{n_I^{\,2}(t)} + \overline{n_Q^{\,2}(t)}\end{aligned} \tag{0.48}$$

By symmetry, $\overline{n_I^{\,2}(t)} = \overline{n_Q^{\,2}(t)}$, and therefore $\overline{\left|n_{\mathbf{E}}(t)\right|^2} = 2\overline{n_I^{\,2}(t)} = 2\overline{n_Q^{\,2}(t)}$.

In other words, the noise power in each component is half the total noise power in the equivalent baseband. Since the one-sided noise spectral density in the equivalent baseband is $4N_0$, and half of this appears in the in-phase component and half in the quadrature component, the one-sided noise spectral density in the $I(t)$ and $Q(t)$ components must be $N_{0I} = N_{0Q} = 2N_0$ each.

What about the energy per symbol in the $I(t)$ and $Q(t)$ signals? Consider a single symbol waveform $I(t)$. In the equivalent baseband, this implies a signal energy in the real (in-phase) component of the equivalent baseband representation of:

$$E_{I\mathbf{E}} = \int_{-\infty}^{\infty} I^2(t)\, dt \tag{0.49}$$

where $E_{I\mathbf{E}}$ is the signal energy in the $I(t)$ symbol in the equivalent baseband. In the real passband, this same signal contains a total energy:

$$\begin{aligned}E_{sI} &= \int_{-\infty}^{\infty} \left(I(t)\cos\left(\omega_c t\right)\right)^2\, dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} I^2(t)\,dt + \frac{1}{2}\int_{-\infty}^{\infty} I^2(t)\cos\left(2\omega_c t\right)dt\end{aligned} \tag{0.50}$$

Using a similar argument to one used before: if $I(t)$ changes much more slowly than $\cos(2\omega_c t)$, which it will if the bandwidth of the passband signal is much smaller than the carrier frequency, the second term will tend to average out to zero, and we're left with:

$$E_s = \frac{1}{2}\int_{-\infty}^{\infty} I^2(t)\,dt = \frac{1}{2}E_I \tag{0.51}$$

Applying the baseband matched filter theory to the $I(t)$ received signal then gives:

$$S_{I\mathbf{E}} N_{I\mathbf{E}} R_{opt} = \frac{2E_{I\mathbf{E}}}{N_{0I}} = \frac{4E_s}{2N_0} = \frac{2E_s}{N_0} \tag{0.52}$$

This is exactly the same result that a matched filter would predict for $I(t)$ if it was a real baseband signal, and had not been up-converted (raised in frequency) at the transmitter and then down-converted (lowered in frequency) at the receiver. Exactly the same result is true for the $Q(t)$ baseband signal as well.

This is an important and often useful point: we can consider the in-phase $I(t)$ and quadrature $Q(t)$ baseband components as having the same amount of noise as if each was a baseband signal, and for the purposes of bit error rate calculation treat $I(t)$ and $Q(t)$ as baseband signals that are both received with a maximum instantaneous signal to noise ratio of $2E_s/N_0$, where in this case E_s is the real baseband energy per symbol in the corresponding $I(t)$ or $Q(t)$ baseband component, and N_0 is the real one-sided noise spectral density.

1.6.1 Noise in the Equivalent Baseband: One Final Result

Often, the noise in the $I(t)$ received signal and the noise in the $Q(t)$ signal both have Gaussian distributions. If the noise component in the real (in-phase) signal has a standard deviation of σ, then the normal (Gaussian) probability density function of this noise is:

$$p(i) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{i^2}{2\sigma^2}\right) \tag{0.53}$$

and in the quadrature direction:

$$p(q) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{q^2}{2\sigma^2}\right) \tag{0.54}$$

where i is the noise value in the in-phase direction, and q the noise value in the quadrature direction. We could write this noise in equivalent baseband form as:

$$n_{\mathbf{E}} = i + jq \tag{0.55}$$

The two-dimensional probability density function[8] (the probability density of the noise at the point $(i + jq)$) can be written:

$$p(i,q) = p(i)\,p(q) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{q^2 + i^2}{2\sigma^2}\right) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) \tag{0.56}$$

where $r = \sqrt{i^2 + q^2}$ is the distance from the origin to the point (i, q). This gives the important result that the probability of the equivalent baseband noise value having a value with an amplitude r is a function of the amplitude only, and not of the phase of the noise term.

[8] The two-dimensional probability density function $p(i, q)$ is defined so that $p(i, q)\, di\, dq$ is the probability of having an i-value between i and $i + di$, and a q-value of between q and $q + dq$.

There's a very important consequence of this result: the probability that adding noise to a signal moves the equivalent baseband representation of the signal a certain distance *r* in any one given direction is:

$$p(d) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{r^2}{2\sigma^2}\right) \quad (0.57)$$

and this result is the same, no matter which direction the point is moved in. It could be along the real (in-phase) axis, or along the imaginary (quadrature) axis, or at an angle of 45-degrees to the real axis, or any other direction, it doesn't matter. The chances of noise moving a signal a certain distance in the given direction is always the same, it doesn't matter which direction you're moving in.

1.7 Key Points

- Any passband signal can be considered in terms of an equivalent baseband signal, where the equivalent baseband signal is a complex signal, with amplitude equal to the amplitude of the passband signal, and phase equal to the difference in phase between the passband signal and the carrier.

- This complex equivalent baseband signal can be divided into two independent signals: the real and imaginary parts of the signal (known as the in-phase $I(t)$ and quadrature $Q(t)$ components respectively).

- The in-phase and quadrature signals behave in noise as if they were baseband signals: both are received after an optimum filter with a maximum signal to noise ratio of $2E_s/N_0$.

- The effect of noise in the equivalent baseband is to move the point representing the passband signal, and the probability that the noise moves this point a distance *d* in any direction is:

$$p(r) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{r^2}{2\sigma^2}\right)$$

- Passband signals can be easily generated by using the in-phase and quadrature component signals, using the in-phase signal to modulate the carrier wave directly, and the quadrature signal to modulate a copy of the carrier wave phase-shifted by one-quarter of a cycle (hence the name 'quadrature'), then adding these modulated signals together.

- Recovery of the in-phase and quadrature signals can also be easily achieved by multiplying the incoming signal by a copy of the carrier wave, and the carrier wave offset by a quarter signal respectively, then low-pass filtering.

1.8 Tutorial Questions

1) At one particular time, the signal received at a receiver is $\pi/4$ radians behind (in terms of phase) the carrier oscillator in the receiver, and of amplitude one. What are the *I*- and *Q*-components of this signal?

*2) A wireless channel has a delay of exactly 1.205 μs. If the carrier frequency is 450 MHz, what is the impulse response of this channel in the equivalent baseband?

***3) For real baseband signals, the output of a filter $y(t)$ in the time domain is the convolution of the input signal $x(t)$ and the filter's impulse response $h(t)$. What about the equivalent baseband domain? What is the output of a filter in the equivalent baseband $y_e(t)$ given the equivalent baseband signal $x_e(t)$, and the equivalent baseband impulse response $h_e(t)$?

*4) Prove that the effective optimum instantaneous equivalent baseband signal to equivalent baseband noise ratio in an equivalent baseband system is E_s/N_0, where E_s is the real passband energy per symbol, and N_0 is the real one-sided passband noise power spectral density.