1 GSW… Fading Distributions

Unless you're very lucky (and even if you are, you won't be this lucky for very long), any mobile channel you try and use will not consist of just one path from the transmitter to the receiver. There could be (and often are) thousands of them. Particularly at mobile devices in urban environments, energy can arrive from just about any direction, having bounced off buildings, hills, trees, the ground, passing cars, other people, donkeys, bicycles, and just about anything else that happens to be around at the time[1].

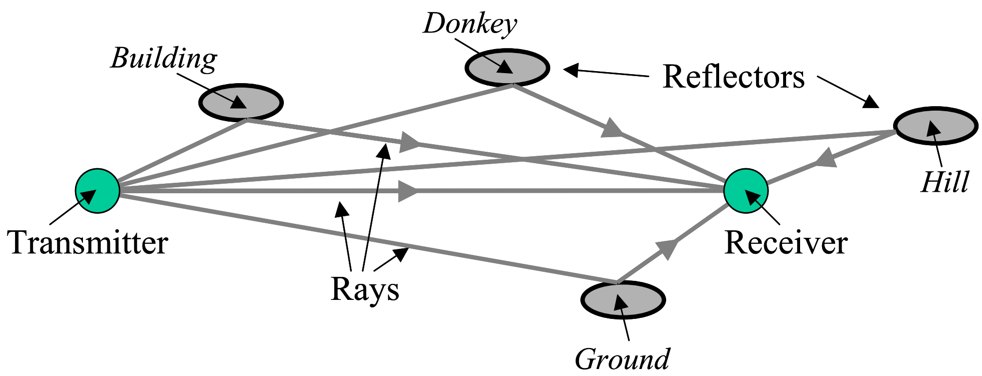

Figure 1-1 Example of a Multipath Channel

All these rays travel different distances before arriving at the received, and hence arrive with different phases and different delays. Since all these reflecting surfaces (except perhaps the hills and the ground[2]) are moving, the delays and phases of the energy arriving from these receivers are constantly changing. The result is a channel with a constantly changing impulse response. This causes a lot of problems.

This chapter is about the statistical models most commonly used to describe these channels.

1.1 Review of Some Basics

Before we start, you need to be very comfortable with the use of vectors to represent oscillations.

1.1.1 Individual Rays as Vectors

A single radio path from a transmitter to a receiver can be characterised by an amplitude and a delay. The signal arriving at the receiver along this path is smaller that the transmitted signal[3],

---

[1] Sorry about the picture, I never was any good at drawing things. You'll have to use your imagination, particularly for the donkey, which should look slightly sad, as it's standing in a gloomy place.

[2] Although including the buildings. Tall buildings can sway several centimetres in a high wind, and that's enough of a wavelength at the frequencies used by mobile phones to cause a large change in the phase of the received signal.

[3] It has to be smaller: the receiver can't get more radio energy out of the air than the transmitter put in.

and arrives after a certain time (in this case the distance between the transmitter and the receiver divided by the speed of light). Mathematically, we can write:

$$r_1(t) = A_1 s(t - \tau_1) \tag{0.1}$$

where $s(t)$ is the transmitted signal, $r_1(t)$ is the received signal from this path, A_1 is the amplitude of the received signal relative to the amplitude of the transmitted signal, and τ_1 is the time it takes the signal to get from the transmitter to the receiver along this path.

In the case when we're transmitting just a single frequency carrier[4], we can write the transmitted signal as:

$$s(t) = \cos(\omega_c t) \tag{0.2}$$

where I've assumed that the transmitted signal has an amplitude of one for simplicity. Substituting equation (0.2) into equation (0.1) then gives:

$$r_1(t) = A_1 \cos(\omega_c t - \omega_c \tau_1) \tag{0.3}$$

The received signal is just another single frequency cosine wave, with a different amplitude (A_1) and a phase offset of $-\omega_c \tau_1$. We can represent such a wave in terms of a vector, with a length of A_1 and in a direction at an angle $\omega_c \tau_1$ to some arbitrary direction (by convention, to the right). For example, this ray could be represented as the vector as shown below:

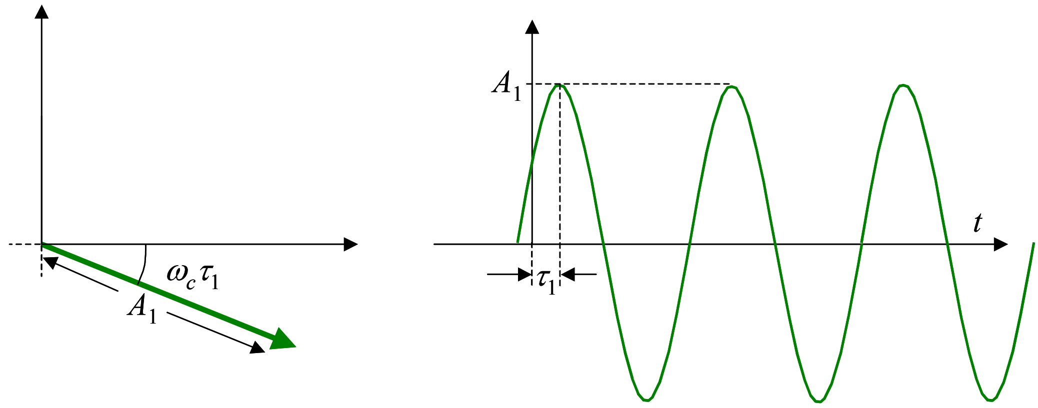

Figure 1-2 Representing Rays as Vectors

A few more examples:

[4] Sometimes known as *continuous wave* or *CW* transmission.

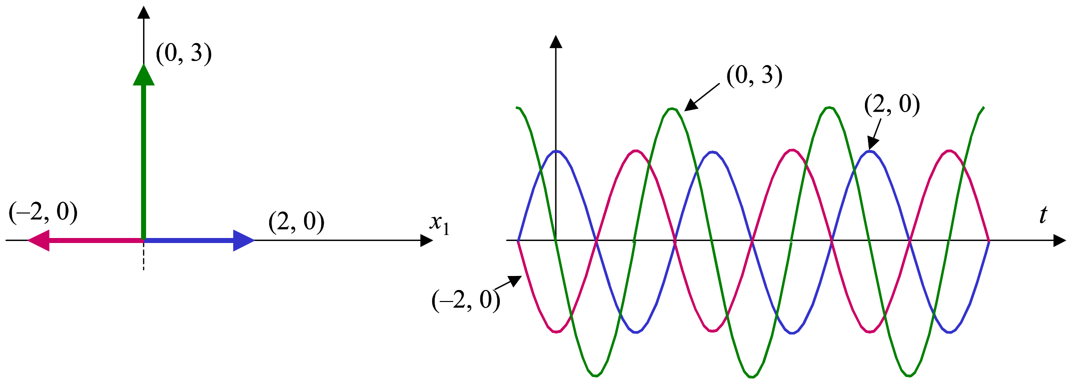

Figure 1-3 More Examples of Representing Rays as Vectors

The whole channel is the sum of a very large number of these rays, we can write:

$$r(t) = \sum_i A_i s(t - \tau_i) \tag{0.4}$$

where $r(t)$ is the total received signal, the sum of all the received signals arriving from all the possible paths.

1.1.2 The Sum of Individual Rays as Vector Addition

To work out the resultant signal from receiving all of these rays, all we need to do is add up the vectors. For example:

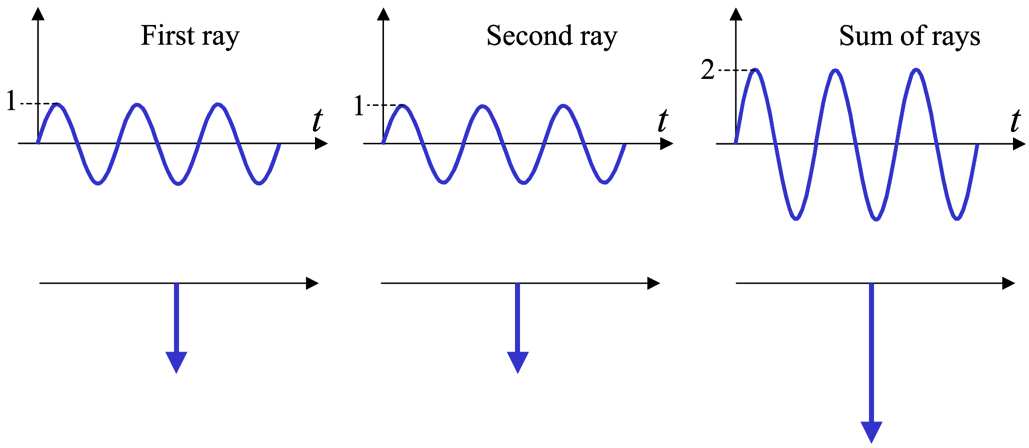

Figure 1-4 Adding Vectors to Add Cosine Waves

This is an example of *constructive interference*: the final resultant signal is larger than either of the two component signals. We can equally get *destructive interference*, when adding together two component rays tends to decrease the amplitude of the resultant. In an extreme case, this can reduce the amplitude of the received wave to zero:

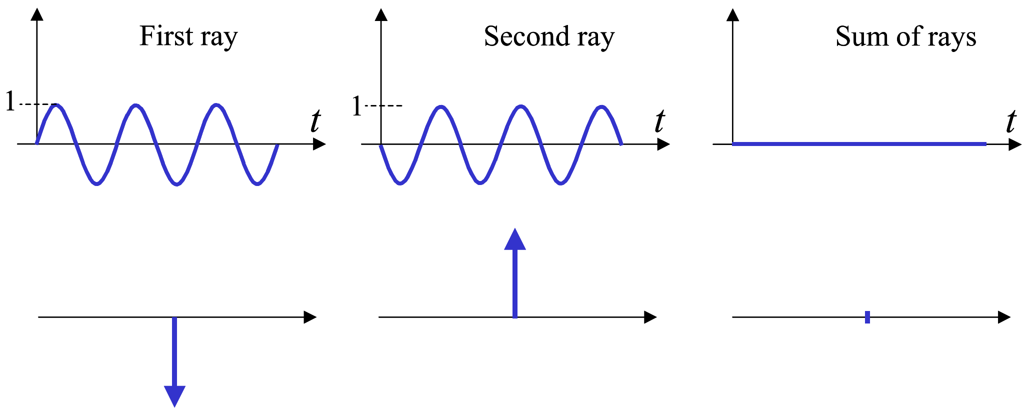

Figure 1-5 Adding Vectors: Destructive Interference

1.1.3 A Simple Example: A Man Walking Away From a Wall

Consider the case of a man walking towards a wall. He receives two rays: one direct from the transmitter, and one reflected off the wall. As he wants, the distance travelled by the ray on the direct path decreases, and the distance travelled by the ray on the reflected ray increases. The result is that the vector representing the direct path continually increases in phase, but the vector representing the reflected ray continually decreases in phase:

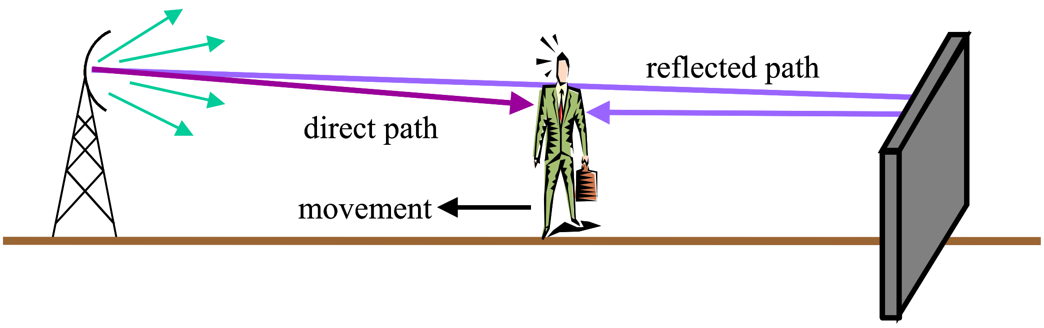

Figure 1-6 A Man Walking Away From a Wall

In terms of the vector representation of these rays, the vector diagram looks like this:

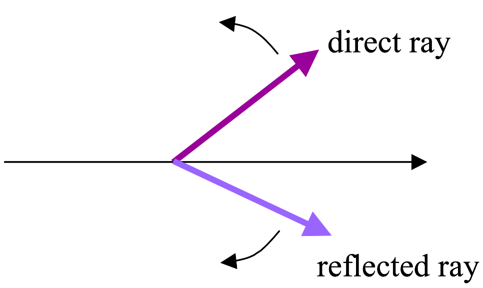

Figure 1-7 Vector Representation of Direct and Reflected Rays

You'll notice that at certain times (or equivalently, certain locations), the two rays will end up pointing in the same direction, and the resultant signal at the man will be very large: this is an example of constructive interference, and provides a large receive signal. Equally, at other times, the two rays will end up pointing in exactly opposite directions, and the resultant signal at the man will be very small; this is known as a *fade*.

This sort of thing is very common. As people move around the environment (or equivalently the environment moves around them: exactly the same effect could have been achieved in the last example by having the man stand perfectly still and then moving the wall), the amplitude of the mobile radio signals they receive is changing all the time.

1.1.4 The Probability Distribution of the Two-Ray Model

We can work out the probability distribution of the amplitude of the received signal for this two ray model, provided we make a couple of simple assumptions: that the phase angle between the two rays has a uniform distribution between –p and p (in other words it's equally likely to have any value, exactly the case if the man is walking at a constant speed), and that the amplitude of the two rays do not change with time (accurate provided the man is a long way from the transmitter, so that the fact that's he's moving closer to the transmitter all the time doesn't result in a significant increase in the power of the direct ray).

This isn't a very useful case for mobile radio, since real situation are usually far more complex than this with many more than two rays arriving, so I'll leave the derivation to the problems, and just give the answer here:

$$p(A) = \frac{2A}{\pi\sqrt{4D^2R^2 - \left(D^2 + R^2 - A^2\right)^2}} \qquad (0.5)$$

where D is the amplitude of the direct ray, and R is the amplitude of the reflected ray.

1.2 Rayleigh Fading

The most common model used for determining the probability of a fade in real mobile radio channels is the Rayleigh distribution. It's a very easy distribution to use, it reflects the observed behaviour of many real channels very well, and it derives from a simple physical model. (All of which is great: we don't often get all three of those advantages together.)

The physical model used to derive the Rayleigh distribution assumes that there is very large number of different reflecting surfaces, randomly distributed around the receiver. This implies a large number of rays arriving at the receiver, all of about the same magnitude (or at least with no one large ray bigger than all the others), and with phases that are equally likely to take any value: mathematically we could say that the phases of the rays has a uniform distribution between $-\pi$ and π, and the phase of each ray is entirely independent of the phases of every other ray.

In this case, we can use the Central Limit Theorem to derive the form of the distribution. Consider: we have a large number of rays, each of which is equally likely to appear in any direction. These are represented by a large number of vectors, of random lengths, pointing in random directions:

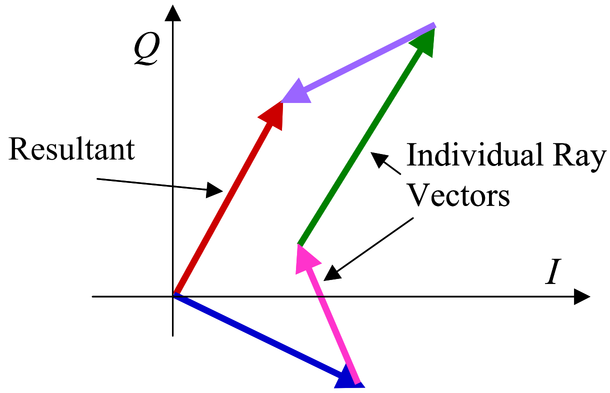

Figure 1-8 Adding Up Random Vectors

The signal finally received by the receiver (the *resultant*) is then represented by the sum of all these random vectors. You might have noticed that I gave the axes names in the last diagram: I called them the *I*-axis and the *Q*-axis. It doesn't matter why I chose those names[5], the important point here is that we can separate each of these vectors representing individual rays into two components: a component along the *I*-axis, and a component along the *Q*-axis:

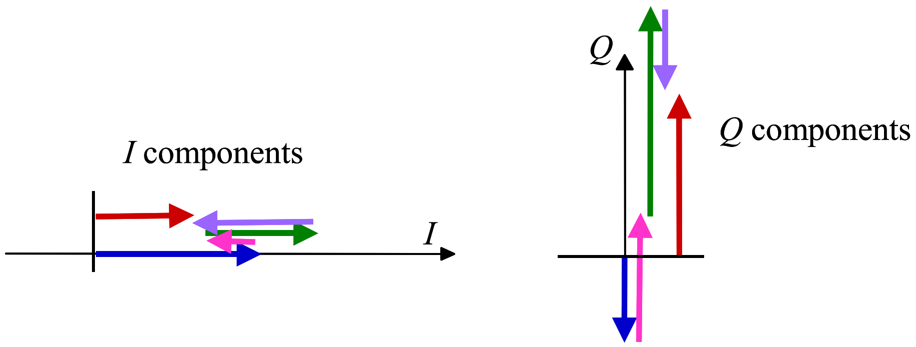

Figure 1-9 *I* and *Q* Components of Individual Rays

The vector representing the final resultant is then the sum of the *I*-components and the *Q*-components. Since they are all vectors in one direction, each of these components can be represented by a single scalar number, which comes from a random distribution with a mean value of zero.

It has to have a mean value of zero, since we assumed that the phases of the these rays were equally likely to be in any direction, and therefore the probability of a positive I-component (one pointing to the right in the diagram above) must be the same as the probability of a negative I-component (pointing to the left); so the mean value must be zero.

The Central Limit Theorem states that the probability distribution of the sum of a large number of independent random numbers has a Gaussian distribution. It doesn't matter what the probability distribution of the individual random numbers is, the probability distribution of the sum of them will be a Gaussian. Here, this implies that the probability distribution of the I-

[5] If you want to know, they stand for In-phase and Quadrature. See the chapters on passband modulation schemes for more details about these ideas.

component of the resultant will be a zero-mean Gaussian, and the probability distribution of the Q-component of the resultant will also be a zero-mean Gaussian.

1.2.1 Deriving the Rayleigh Distribution

Let the I-component of the resultant received signal be I, and the Q-component be Q. Then, the probability that the resultant has lies at the point (I, Q) is:

$$\begin{aligned} p(I,Q) = p(I)\,p(Q) &= \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{I^2}{2\sigma^2}\right)\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{Q^2}{2\sigma^2}\right) \\ &= \frac{1}{2\pi\sigma^2}\exp\left(-\frac{I^2+Q^2}{2\sigma^2}\right) \end{aligned} \quad (0.6)$$

Now if the amplitude of the resultant vector is A, then Pythagorus' theorem gives:

$$A^2 = I^2 + Q^2 \quad (0.7)$$

and so:

$$p(I,Q) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{A^2}{2\sigma^2}\right) \quad (0.8)$$

If we want the probability of the resultant amplitude as a function of the amplitude of the signal A, all we need to do is integrate this value over all possible values of I and Q which have the same amplitude.

Consider: the probability of the resultant being in a small area between I and $I + \delta I$ and between Q and $Q + \delta Q$ is:

$$p(I,Q)\,dI\,dQ = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{A^2}{2\sigma^2}\right)dI\,dQ \quad (0.9)$$

(that comes straight from the definition of a probability density function). The probability that the resultant has an amplitude of between A and $A + \delta A$, is likewise, $p(A)\ \delta A$.

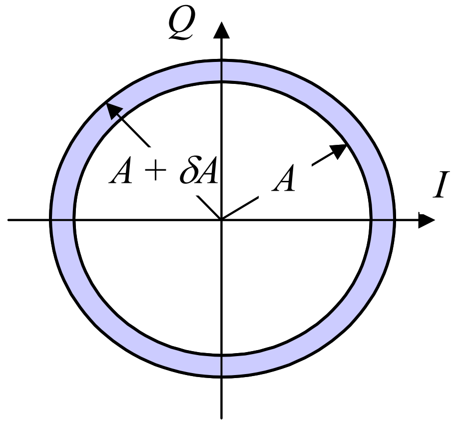

Figure 1-10 Integrating to Find the Rayleigh Distribution

If we add up the probability that the resultant lies in a ring at a distance between A and $A + \delta A$ from the origin, then we get:

$$p(A)dA = \int_{ring} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{A^2}{2\sigma^2}\right) dI\,dQ \tag{0.10}$$

Within this narrow ring, we can consider the value of A to be constant, so we can take the terms outside the integral sign, and write:

$$p(A)dA = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{A^2}{2\sigma^2}\right) \int_{ring} dI\,dQ \tag{0.11}$$

and the area of the ring is just $2\pi A\ \delta A$, so:

$$p(A)dA = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{A^2}{2\sigma^2}\right) 2\pi A\,dA \tag{0.12}$$

which gives:

$$p(A) = \frac{A}{\sigma^2} \exp\left(-\frac{A^2}{2\sigma^2}\right) \tag{0.13}$$

and this is one form of the Rayleigh distribution. It's not the most useful form, since σ (the standard deviation of the components) isn't a very good parameter to use to specify the distribution: it's not directly measurable. If you determine the mean power in the Rayleigh distribution using the above formula, you'll find that:

$$E\left\{A^2\right\} = \int_0^\infty A^2 p(A)\,dA = \int_0^\infty \frac{A^3}{\sigma^2} \exp\left(-\frac{A^2}{2\sigma^2}\right) dA \tag{0.14}$$

which after rather a lot of tedious calculus and algebra[6] gives the very simple result:

$$E\left\{A^2\right\} = 2\sigma^2 \tag{0.15}$$

Writing the mean power as χ then allows a more convenient form of the Rayleigh distribution:

$$p(A) = \frac{2A}{\chi} \exp\left(-\frac{A^2}{\chi}\right) \tag{0.16}$$

where χ is the mean power in the received signal.

[6] Or you could just note that the power is proportional to the square of the amplitude, so we can just add the powers in the I and Q components, and these are Gaussians, each of which has a mean power of σ^2.

1.2.2 Deriving the Exponential Distribution

The Rayleigh distribution is a probability distribution of the amplitude in the received signal. It's usually easier to work out problems in terms of the power of the received signal: you might, for example, want to know how often the power fades to less than 20 dB below the mean power. For problems like this, it's easier to use the corresponding distribution for power.

The question then: if a fading signal has an amplitude distribution given by the Rayleigh distribution, what distribution describes the probability distribution of the received power? This is quite easy to work out, and gives a very easy answer.

The probability of the amplitude being between A and $A + \delta A$ is:

$$p(A)\,dA = \frac{2A}{\chi}\exp\left(-\frac{A^2}{\chi}\right)dA \tag{0.17}$$

this must be equal to the probability of the power being between P and $P + \delta P$, where $P = A^2$:

$$p(P)\,dP = p(A)\,dA \tag{0.18}$$

Therefore:

$$p(P) = p(A)\frac{dA}{dP} \tag{0.19}$$

and since simple differentiation of $P = A^2$ gives $dP = 2AdA$:

$$p(P) = \frac{2A}{\chi}\exp\left(-\frac{A^2}{\chi}\right)\frac{1}{2A} = \frac{1}{\chi}\exp\left(-\frac{P}{\chi}\right) \tag{0.20}$$

This is a negative exponential distribution, and it's even easier to work with than the Rayleigh distribution. Integrating this one is (almost) trivial.

1.2.3 Example of Rayleigh Fading

Suppose I have a Rayleigh fading signal, with a mean received power of –80 dBm. What is the probability that at any given time, the received signal will have a power of less than -100 dBm?

I'll use the negative exponential distribution of the received power here, since it makes the maths much easier. The mean received power is $\chi = -80$ dBm $= 10^{-8}$ mW, and we want the probability that the received signal is less that –100 dBm, so all we need to do is integrate the probability density function from zero up to –100 dBm:

$$\int_0^{10^{-10}} \frac{1}{10^{-8}}\exp\left(-\frac{P}{10^{-8}}\right)dP = \left[-\exp\left(-\frac{P}{10^{-8}}\right)\right]_0^{10^{-10}} = 1-\exp\left(-\frac{10^{-10}}{10^{-8}}\right) \tag{0.21}$$

which works out to be just under 1%.

You might find that a bit surprising? What it means is that if we had a receiver with a sensitivity 20 dB lower than the mean received power (so that it would work perfectly well with only 1% of the mean received power level) then this receiver will only work around 99% of the time.

You might also note the apparent co-incidence here: the link fails 1% of the time, with a receiver of sensitivity 1% of the mean received power. It's not co-incidence. Suppose you wanted to the link to fail a proportion x of the time, where x is a small number. Then, the sensitivity of the receiver P_s must be:

$$x = 1 - \exp\left(-\frac{P_s}{\chi}\right) \tag{0.22}$$

so:

$$P_s = -\chi \ln\left(1 - x\right) \tag{0.23}$$

and for small values of x, we can approximate $\ln(1 - x) \approx -x$, which gives:

$$P_s \approx \chi x \tag{0.24}$$

It's a useful approximation: for a link to fail only 3% of the time in Rayleigh fading, you need a receiver with a sensitivity 3% of the mean received power. To fail only 0.1% of the time, you need a sensitivity 0.1% of the mean received power (30 dB lower), and so on.

1.2.4 *Properties of the Rayleigh Distribution*

For interest, the Rayleigh distribution looks like this:

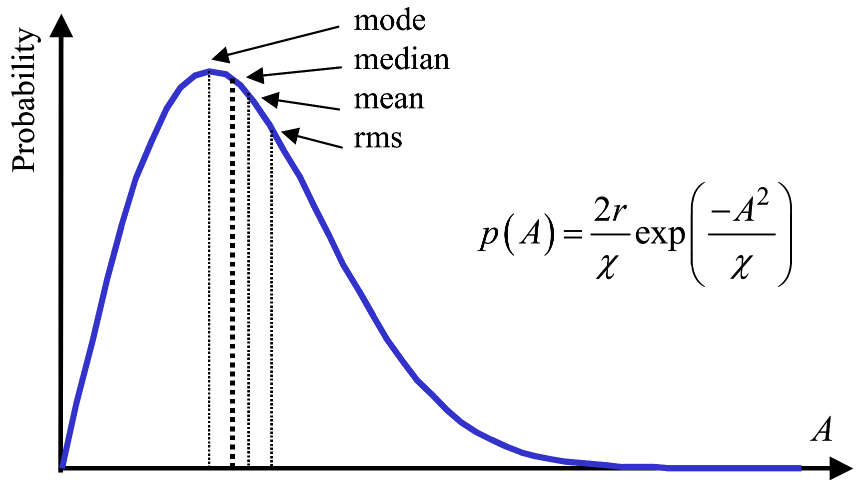

Figure 1-11 The Rayleigh Distribution

For very low values of amplitude (which is often the most interesting part of the distribution, since this is where things stop working), the distribution is almost a straight line.

1.3 Ricean Fading

There's another distribution often used for mobile channel, which can also be derived from a simple physical model. Unfortunately, it's not very simple to derive, and not very easy to work with. It's called the Rice distribution, and it's used when there is a single, large component (perhaps from a line-of-sight path) together with a large number of small random component. We could illustrate it as follows:

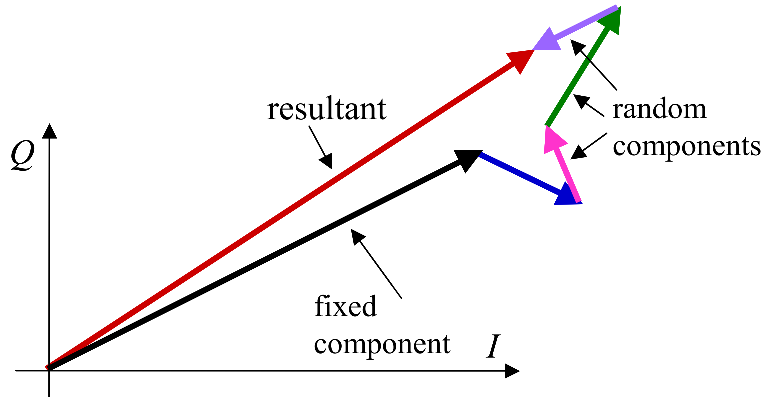

Figure 1-12 Ricean Fading

It's effectively Rayleigh fading added onto a single large fixed component. The derivation is beyond the scope of this book, but for interest's sake, the result is:

$$p(A) = \frac{A}{\sigma^2} \exp\left(-\frac{A^2 + A_s^2}{2\sigma^2} \right) I_0\left(\frac{A_s A}{\sigma^2} \right) \tag{0.25}$$

where A_s is the amplitude of the large fixed component, and $I_0(x)$ is a modified Bessel function of the first kind, with zero order.

Unlike the Rayleigh function, where the only thing you have to know is the mean received power, to use the Rice distribution you need to know both the mean received power in the random component ($2\sigma^2$), and the power in the fixed component (A_s^2). This makes the Rice distribution more awkward to use: you need more information about the particular channel you are concerned with.

The Rice distribution is often characterised in terms of the ratio of these two powers: the k-value is defined as the ratio of the power in the fixed component to the mean power in the random components:

$$k = \frac{A_s^2}{2\sigma^2} \tag{0.26}$$

and the total mean power in the Ricean distribution is the sum of the powers in the fixed and random components:

$$\chi_{ricean} = 2\sigma^2 + A_s^2 \tag{0.27}$$

For small values of k, we can neglect the fixed component, and we just get Rayleigh fading again. For large values of k, the Ricean distribution can be approximated by a Gaussian distribution.

1.3.1 Comparison of Fading Distributions

A comparison between the Rayleigh, Ricean and Gaussian distributions is shown below.

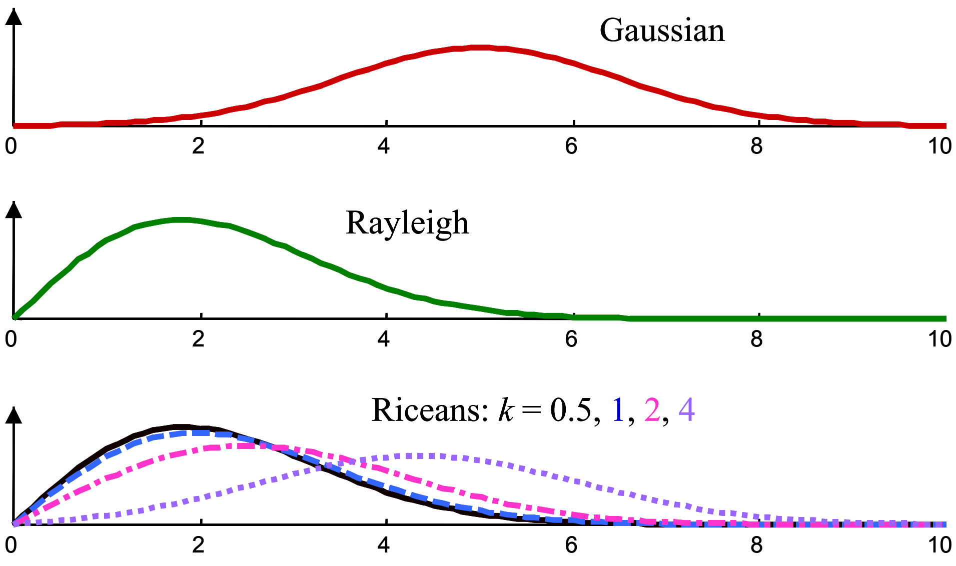

Figure 1-13 Comparison of Fading Distributions

Note that Ricean distributions with a small *k*-value look very like Rayleigh distribution, and with a large *k*-value, they look a lot like Gaussians.

1.4 Other Distributions

From time to time other people propose distributions to use for fading channels, such as the Nakagami distribution, which approximates the Rice distribution but is easier to integrate and can be expressed in dB form:

$$p(A) = \frac{2m^m}{(m-1)!\,\chi^m} A^{2m-1} \exp\left(\frac{-mA^2}{\chi}\right) \qquad (0.28)$$

where χ is the mean power, and m is a shape parameter that can be set according to the k-value of the Rice distribution to be approximated:

$$m = \frac{(k+1)^2}{(2k+1)} \qquad (0.29)$$

(You might notice that setting $m = 1$ gives $k = 0$, and the Rayeigh distribution.)

Despite the mathematical convenience of this, the Rayleigh and Rice distributions are by far the most common and widely used. So much so, in fact, that I don't think I'll bother writing any more about the others.

1.5 Key Points

- The Rayleigh distribution is the most useful and simplest mathematical model used for multipath propagation; it corresponds to the case of a very large number of component rays of similar amplitude arriving with uniformly-distributed phases.

- The Rice distribution is used when there is a single, large, non-fading component to the received signal, as well as a large number of smaller rays with uniformly-distributed phases.

- The Rayleigh distribution implies that for an outage of x % (where x is small), the sensitivity of the receiver is approximately x % of the mean received power.

1.6 Problems

1) A mobile radio link operating over a Rayleigh fading channel uses a receiver with a sensitivity of –90 dBm. If the mean received power is –75 dBm, for what proportion of the time is this link likely to work?

2) A radio link is specified as having to work 99.99% of the time (so-called 'four-nines availability). The power budget suggests the mean receive power is likely to be –75 dBm, but the environment is constantly moving, with heavy scattering and multipath, and no single clear path from the transmitter to the receiver. What receiver sensitivity is likely to be needed?

What about if the link was only require to work 75 % of the time?

3) Derive the probability distribution for the two-ray model given in section 1.1.4. (Hint, you can consider the range of the phase angle between the two rays as going from 0 to π only, since the resultant amplitude for a phase angle of θ is the same as that for $-\theta$.)

4) The Nakagami distribution is a distribution of amplitudes. What is the corresponding distribution of received powers?

5) Derive an expression for the Rice distribution in terms of the mean receiver power and the k-value.

6) Using the Nakagami approximation to the Rice distribution when $k = 3$, what percentage of the time is the received signal power less that 1% of the mean received power? How does this compare to the exact answer using the Rice distribution (this will require some computer help).