1   GSW… Vectors

Mathematically, a vector is a set of real or complex numbers arranged in a linear array. The linear array can be vertical, in which case the vector is known as a *column vector*, for example:

$$\mathbf{a} = \begin{bmatrix} 2 \\ 0 \\ 4 - 3j \end{bmatrix} \tag{0.1}$$

or horizontal, in which case the vector is known as a *row vector,* for example:

$$\mathbf{b} = \begin{bmatrix} 3 & 2 & j & 2 - j \end{bmatrix} \tag{0.2}$$

Note that the elements of vectors can be complex, and vectors can have any number of *elements*. (The column vector **a** has three elements; the row vector **b** has four elements.)

1.1 Uses of Vectors

In communications engineering (and therefore in this book), vectors are used for five distinct purposes: representing points in space; representing the distance between two points in space; representing terms in a set of simultaneous equations; representing the magnitude and direction of electric and magnetic fields in space; and representing the phase and amplitude of oscillating signals. In the first three cases the vectors can have any number of elements; in the fourth case they always have three (since we live in three-dimensional space), and in the fifth case, they always have two.

1.1.1   Vectors as Points in Space

You might have noticed that in the paragraph above, I hinted that vectors representing points in space could have any number of elements. Obviously, to represent a point in the familiar three-dimensional space we live in, vectors must have three elements; but the concept of a vector space is more generally useful, and is often extended to any number of elements.
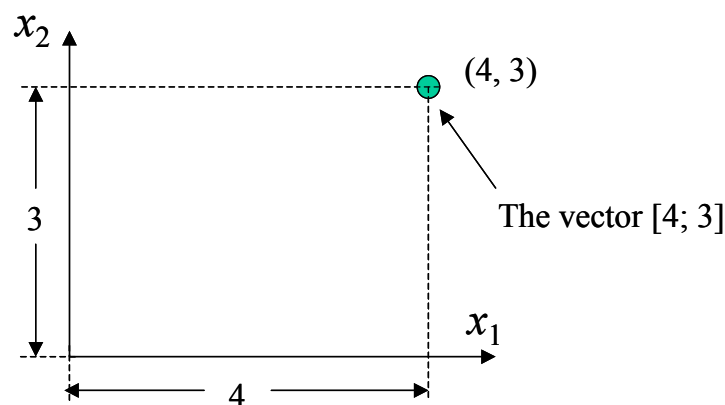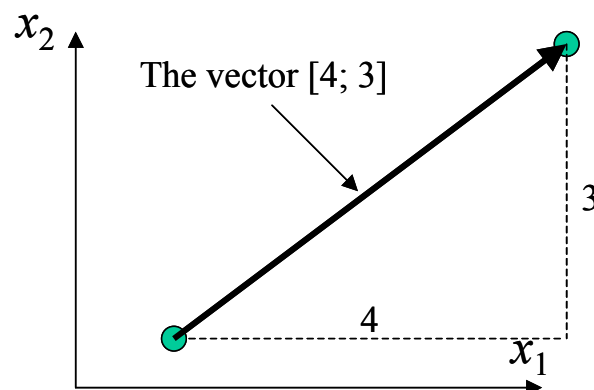

Figure 1-1 The Vector as a Point in Space

For example, a vector with a single element (a scalar) can represent a point on a line: the most familiar example would be the line of real numbers. A vector with two elements can represent a point on a plane (see, for example, the vector[1] [4; 3] in Figure 1-4. A vector with four elements can represent a point in four-dimensional space, and so on. Thinking about vectors in this way often provides some useful geometric insights into the solutions of simultaneous equations (see the chapter on Linear Algebra for some good examples).

1.1.2   Vectors as Distances Between Points in Space

Closely related to the previous use of vectors as points in space, is the use of vectors to represent the distance between two points, as shown in Figure 1-2.

Figure 1-2 The Vector as a Distance Between Points in Space

If the first point (the start of the vector) is the origin, then the end of the vector will be at the point represented by the vector. Often in communications engineering we're interested in maximising or minimising the distance between two points; there are lots of examples of this throughout the rest of the book.

1.1.3   Vectors as the Terms in Simultaneous Equations

Very often in communication engineering, we're required to solve sets of linear simultaneous equations, such as:

$$4 = 3x_1 - x_2$$
$$3 = 5x_1 + 2x_2$$

(0.3)

and these are often conveniently written in the form of vectors and matrices[2]. In this example, we'd get:

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[1] Note that [4; 3] is a column vector. The ";" indicates the end of one row and the start of another. This is the notation is the one MATLAB uses, so I'll adopt it here. The vector [4 3] would be a row vector. In this book, all vectors are column vectors unless otherwise indicated.

[2] If you're not sure how this form works, see the chapter on Matrices.

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{0.4}$$

where the unknown terms appear as the elements in a column vector. They could, for example, be used to represent the outputs of different antennas, or samples of the same signal separated in time. The important point is that the elements in the vector all represent the same sort of thing, and they're all measured in the same units.

1.1.4   Vectors as Fields in Three-Dimensional Space

The electric field at any point in space has a value (measured in Volts per metre) and a direction (the direction of the force experiences by a small positive charge placed at the point in space). Electric field can therefore be represented mathematically by associating a vector with every point in space. This creates a *vector field*.

The same is true for magnetic fields and gravitational fields, although the latter is of less interest in communications engineering (except perhaps when calculating satellite orbits).

Since we live in three-dimensional space, these vectors always have three elements: the components of the field along the co-ordinate axes being used.

1.1.5   Phasors: Vectors as Oscillations

Two-dimensional vectors can be used to represent sinusoidal oscillations of any given phase and amplitude, provided you know the frequency. Vectors used in this way are known as *phasors*. They're often used in the analysis of linear systems, since linear systems can't change the frequency of signals; they just change the amplitudes and phases.

It's very similar to how oscillations are represented by complex numbers. A quick reminder[3]: there are two ways to represent an oscillation of known frequency but unknown amplitude and phase using complex numbers. The first is to define the real, oscillating waveform as the real part of the product a complex number and a complex oscillation:

$$x(t) = \Re\left\{ x_e(t) \exp\left(j\omega_c t \right) \right\} \tag{0.5}$$

and the second is to use two complex oscillations, with the sum always being positive[4]:

$$x(t) = x_f(t) \exp\left(j\omega_c t \right) + x_f^{\,*}(t) \exp\left(-j\omega_c t \right) \tag{0.6}$$

A complex number represents a point in a two-dimensional plane of complex numbers (the Argand diagram), and a two-dimensional vector can represent a point in a plane. The vector representing the point $x_e(t)$ is the phasor representation of this oscillation.

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[3] See the chapter on Complex Numbers for more details.

[4] Note that in this case, $x_e(t) = 2\,x_f(t)$. I've used the suffix 'e' for one, since it the form used by the equivalent baseband representation, and the suffix 'f' for the other, since it is the form used by the Fourier transform.

The idea is illustrated in Figure 1-3 below: the sinusoidal waveform represented by the vector [4; 3] has an amplitude of five, and a phase angle of atan(3/4) = 36.9 degrees compared to a reference sinusoid of the same frequency: the reference sinusoid (shown as a dotted line) having an amplitude of one, and a phase of zero.
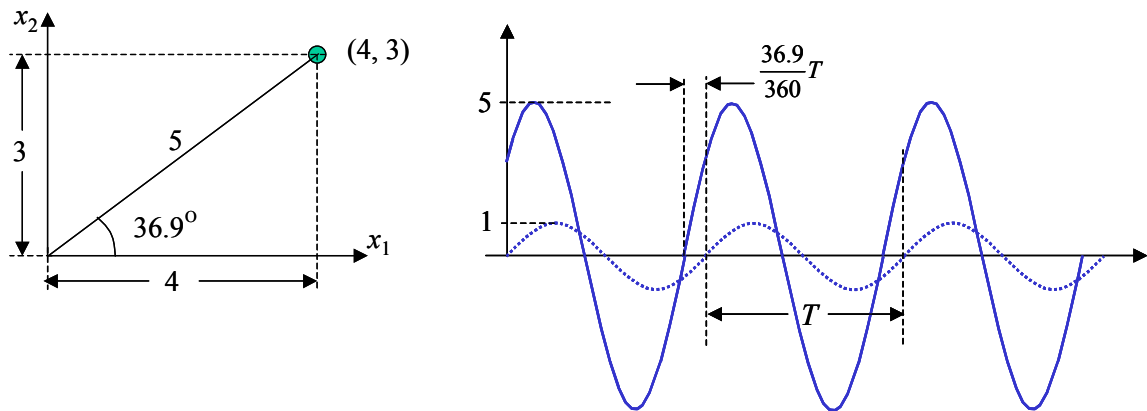

Figure 1-3 Phasors: The Vector as an Oscillation

1.2 Vector Arithmetic

Adding points in space doesn't make a lot of sense, but adding the distance between points does. Suppose two vectors represent different distances travelled in different directions: the addition of the two vectors is just the total distance from the starting point if you travel the distance and direction indicated by the first vector, followed by the distance and direction indicated by the second:
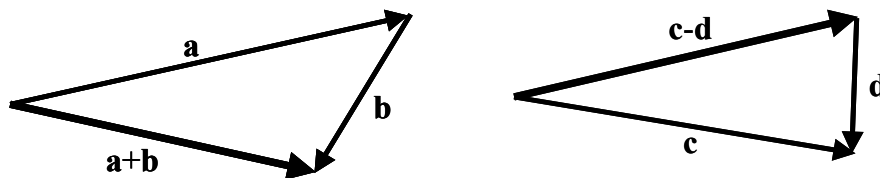

Figure 1-4 - Adding and Subtracting Vectors

While adding points in space together might not make much sense, subtracting them does: the vector formed by the subtraction of two points in space is a vector representing the distance between these points. Subtraction, as you can also see from the above diagram, can be readily defined in similar terms, so that $(\mathbf{c} - \mathbf{d}) + \mathbf{d} = \mathbf{c}$, as you might expect.
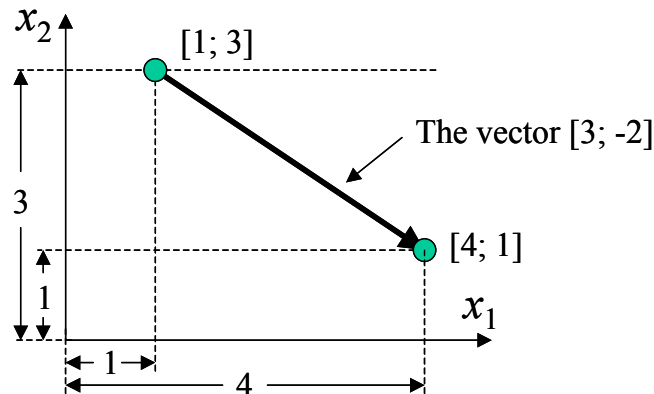
Figure 1-5 Subtraction as the Distance Between Points

In terms of the components of the vectors, this is the same thing as just adding up the individual components. For example, consider a vector that represents a distance 2 meters east, 3 meters north, and 5 meters up. This could be written:

$$\mathbf{e} = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix} \tag{0.7}$$

And another vector that represents a distance 3 meters west, 2 meters south, and 3 meters down:

$$\mathbf{f} = \begin{bmatrix} -3 & -2 & -3 \end{bmatrix} \tag{0.8}$$

If you travelled the first distance, and then the second distance, you would end up having travelled a distance:

$$\mathbf{e} + \mathbf{f} = \begin{bmatrix} -1 & 1 & 2 \end{bmatrix} \tag{0.9}$$

which is one meter south, one meter east and two meters above where you started. Vectors are always added and subtracted like this[5].

Since vector addition and subtraction are just element-by-element additions and subtractions, they share the properties of addition and subtraction of scalar numbers, for example:

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= \mathbf{b} + \mathbf{a} \\ \mathbf{a} - \mathbf{b} &= -(\mathbf{b} - \mathbf{a}) \\ \mathbf{a} - (\mathbf{b} + \mathbf{c}) &= (\mathbf{a} - \mathbf{b}) + \mathbf{c} \end{aligned} \tag{0.10}$$

In terms of vector fields, addition gives the resultant field when two component fields, perhaps the electric fields due to different charges, are combined. In terms of phasors, the sum of two vectors gives a representation of the sum of the two oscillations, for example:

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[5] If you're using a multi-valued quantity that doesn't behave like this in real life, then you can't use a vector to represent it. Also, note that this implies you can only add two vectors if they have the same number of elements.
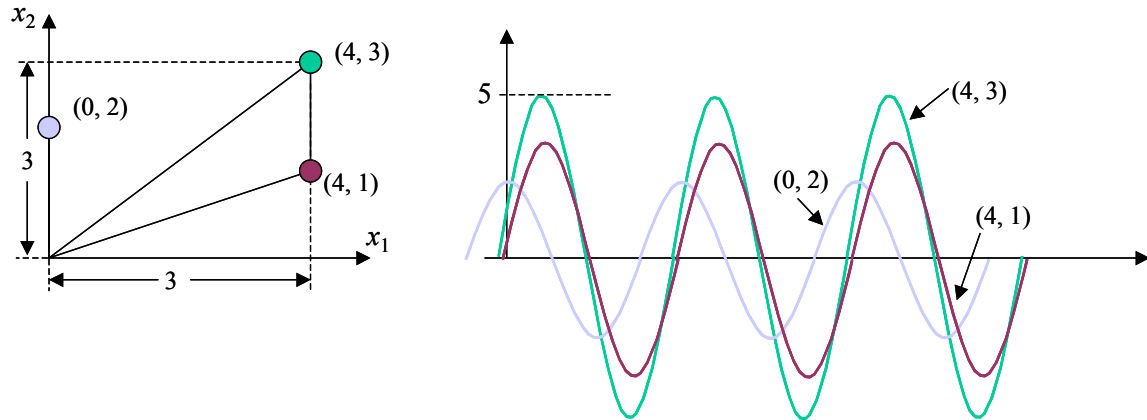
Figure 1-6 Adding Phasors

This reflects the fact that any oscillation of a particular angular frequency ω can be expressed as the sum of a cosine wave and a sine wave:

$$A\cos(\omega t + \phi) = x_1 \cos(\omega t) + x_2 \sin(\omega t) \tag{0.11}$$

and the amount of cosine wave (x_1) and sine wave (x_2) required are represented by the two components of the vector. The result from adding two (or any number) of sinusoidal waveforms can be determined by simply adding the two (or any number) of two-element vectors.

1.3 Some More Notation

Again thinking of vector in terms of a line, any vector **a** has an *amplitude*, often written as ||**a**||, which is the length of the line, and also a *direction*. A *unit vector* (that is, a vector with an amplitude of one) in the direction of **a** is written as $\hat{\mathbf{a}}$. A three-dimensional vector representing a distance can therefore be expressed as:

$$\mathbf{a} = a_x\,\hat{\mathbf{x}} + a_y\,\hat{\mathbf{y}} + a_z\,\hat{\mathbf{z}} \tag{0.12}$$

where $\hat{\mathbf{x}}$ is a unit vector in a direction along the x-axis, $\hat{\mathbf{y}}$ a unit vector along the y-axis, and $\hat{\mathbf{z}}$ a unit vector along the z-axis, and a_x, a_y and a_z are the scalar components of this vector along these three axes. In more familiar co-ordinate notation, we could just refer to the vector as (a_x, a_y, a_z), or in the notation of vectors, as $[a_x \quad a_y \quad a_z]$. They all mean the same thing.

This vector has an amplitude given by applying Pythagorus' theorem to the problem[6]:

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[6] If you're not familiar with this notation: ||**a**|| is the amplitude of the vector **a**, sometimes called the *Euclidean norm* of **a**. In general, when the elements of a are complex, it's defined by:

$$\|\mathbf{a}\| = \sqrt{\left|a_x\right|^2 + \left|a_y\right|^2 + \left|a_z\right|^2}$$

but here I'm assuming that all the elements of **a** are real, so I'm not bothering to take the modulus of each element. (For more on vectors with complex elements, see later on in this chapter.)

$$\|\mathbf{a}\|^2 = a_x{}^2 + a_y{}^2 + a_z{}^2 \tag{0.13}$$

Finally, although there are two ways of writing a vector (as a *row vector*, e.g.

$$\mathbf{a} = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}$$

or as a *column vector*, e.g.

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \tag{0.14}$$

), in terms of the distance, point or field they represent, they are exactly the same thing. The only reason to distinguish them is when you want to start multiplying them together.

1.4 Vector Multiplication

Multiplying a vector by a scalar is easy: you just multiply each element of the vector by the scalar. This results in a vector in the same direction[7], but a different amplitude.

$$\mathbf{a} = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}$$
$$-2\mathbf{a} = \begin{bmatrix} -4 & 2 & -6 \end{bmatrix} \tag{0.15}$$

However, multiplying two vectors together is a bit strange. There are three ways to multiply two vectors together, all of which give different answers. The most useful answer depends on what the vectors are representing.

1.4.1   Vector Multiplication: The Dot Product

The first way to multiply vectors together is known as the *dot product*, *scalar product* or *inner product*, and is written with a dot:

$$x = \mathbf{a}.\mathbf{b} \tag{0.16}$$

It's calculated by taking the product of the individual elements of **a** with the corresponding elements of **b**, and then adding up all these products of the elements, e.g.:

---

[7] Note that pointing in exactly the opposite direction counts as the 'same direction' with vectors. You could think of it as a vector in the same direction, but with a negative amplitude.

$$\mathbf{a} = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

(0.17)

$$\mathbf{a.b} = \begin{bmatrix} 1 \times 3 & -2 \times -1 & 3 \times -2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -6 \end{bmatrix}$$

(You'll notice that I wrote the first vector **a** as a row vector, and the second vector **b** as a column vector. This is the conventional way of writing the inner product, since it makes the definition of vector inner multiplication consistent with matrix multiplication. More about this in the chapter on Matrices.)

It's called the *scalar product* since it is the only one of the three methods of multiplying vectors that results in a scalar. In terms of vectors as representing points in space, or oscillations, this doesn't make a lot of sense; but when the vectors represent distances, we get a very interesting and useful result. First, note that from Pythagorus' theorem (see equation (0.13)), the square of the length of the vector[8] is:

$$\|\mathbf{a}\|^2 = a_x{}^2 + a_y{}^2 + a_z{}^2 = \mathbf{a.a}$$

(0.18)

(and again, here I'm assuming that the elements of **a** are real). Then, consider two vectors **a** and **b** representing distances in space, and let them both start at the same point:
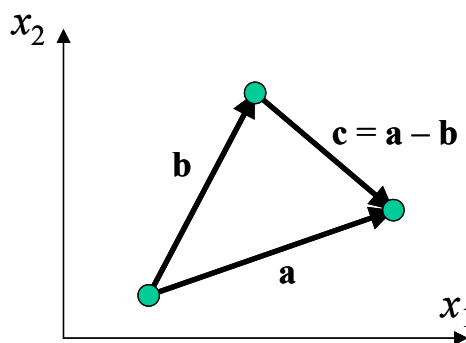
Figure 1-7 Adding Vectors and the Cosine Rule

First, define **c** as the difference between the vectors **a** and **b**:

$$\mathbf{c} = \mathbf{a} - \mathbf{b}$$

(0.19)

Taking the dot product of each side with itself, gives[9]:

---

[8] I'm playing a bit loose with the notation here: strictly speaking, you could object to the notation **a**.**a** since a dot product must be a row vector times a column vector, and **a** can't be both. More properly, I should write $\mathbf{a}^T\mathbf{a}$, where $\mathbf{a}^T$ is the transpose of **a** (the transpose of a column vector is a row vector with the same elements). More about transposes in the chapter on matrices.

[9] Again, I'm using the notation loosely, and I should perhaps write $(\mathbf{a} - \mathbf{b})^T.(\mathbf{a} - \mathbf{b})$, etc, so that $(\mathbf{a} - \mathbf{b})^T$ is a row vector.

$$\mathbf{c}.\mathbf{c} = (\mathbf{a} - \mathbf{b}).(\mathbf{a} - \mathbf{b})$$

$$\|\mathbf{c}\|^2 = \mathbf{a}.\mathbf{a} - \mathbf{a}.\mathbf{b} - \mathbf{b}.\mathbf{a} + \mathbf{b}.\mathbf{b} \qquad (0.20)$$

$$\|\mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\mathbf{a}.\mathbf{b}$$

Compare this with the cosine-rule[10] in the triangle formed by the vectors **a**, **b** and **c**:

$$\|\mathbf{c}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta) \qquad (0.21)$$

and we get the interesting result:

$$\mathbf{a}.\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta) \qquad (0.22)$$

Interpreting the vectors as distances: the dot product of two vectors is equal to the product of the amplitudes of the vectors and the cosine of the angle between them. This is a very useful result when working out some trigonometric problems, especially in three-dimensions; see the problems for an example.

A related result allows the simple calculation of the *projection* of a vector along another vector. The projection of a vector **a** along a vector **b** can be thought of as the length of the shadow cast by vector **a**, assuming that the light is coming in a direction perpendicular to the vector **b**. In other words, a line from the end of the projection of **a** along **b**, to the end of vector **a** is perpendicular to vector **b**. See Figure 1-8:
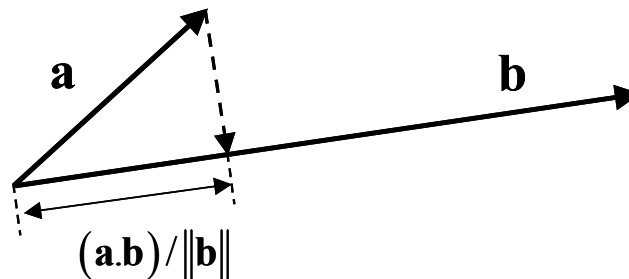
Figure 1-8 - The Projection of a onto b

Using the dot-product, and noting that the projection of **a** onto **b** and the vector **a** form two sides of a right-angled triangle, we can quickly derive that the length of the projection:

$$\|\mathbf{a}\|\cos(\theta) = \frac{\|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta)}{\|\mathbf{b}\|} = \frac{\mathbf{a}.\mathbf{b}}{\|\mathbf{b}\|} \qquad (0.23)$$

When the vector **b** is a unit vector, we get the particularly simple result that the length of the projection is just equal to **a.b**.

Remember, in terms of row and column vectors, the scalar, dot or inner product is what you get when you multiply a row vector by a column vector, but it has to be in that order, with the row vector first, and the number of elements of the two vectors has to be the same. For example:

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[10] See the chapter on Trigonometry if you've forgotten what the cosine rule is all about.

$$\begin{bmatrix} a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \mathbf{a} . \mathbf{b} = a_x b_x + a_y b_y + a_z b_z \qquad (0.24)$$

Multiply them together the other way around (with the column vector first) and you get something completely different.

1.4.2   Vector Multiplication: The Outer Product

Multiply a column vector by a row vector, and you get the *outer product*. This results in a matrix. This time it doesn't matter if the vectors have the same number of elements or not, you can still define an outer product. If you've not come across matrices before, don't worry, see the chapter on matrices for more details.

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \begin{bmatrix} b_w & b_x & b_y & b_z \end{bmatrix} = \begin{bmatrix} a_x b_w & a_x b_x & a_x b_y & a_x b_z \\ a_y b_w & a_y b_x & a_y b_y & a_y b_z \\ a_z b_w & a_z b_x & a_z b_y & a_z b_z \end{bmatrix} \qquad (0.25)$$

1.4.3   Vector Multiplication: The Cross Product

This one only works for three-element vectors representing distances or fields in three-dimensional space. The cross product of two vectors is the area of the parallelogram defined by the two vectors. For example:
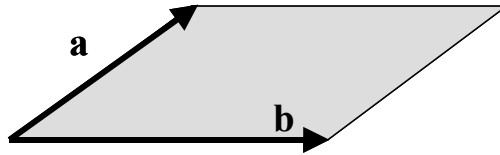

Figure 1-9 - The Vector Cross Product

The area of this parallelogram is given by $\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$, where θ is the angle between the two vectors. The vector cross product has an amplitude given by the dot product of **a** with a vector in the same plane as **a** and **b**, and of the same length as **b**, but at right angles to **b.**

In vector language, areas have a direction (the direction perpendicular to the surface) so the cross product of two vectors is another vector. To determine which direction, the "right-hand rule" is used. Align the first vector with the thumb of your right-hand, and the second vector with your first (index) finger; then bend your second finger at right-angles to the thumb and first finger. The direction your second finger is now pointing in gives the direction of the area. In the case shown above, $\mathbf{a} \times \mathbf{b}$ is a vector going down into the paper away from you, $\mathbf{b} \times \mathbf{a}$ is a vector coming up out of the paper towards you. (Cross-products are written $\mathbf{a} \times \mathbf{b}$ or sometimes as $\mathbf{a} \wedge \mathbf{b}$.)

Again, in Cartesian co-ordinates, the cross product can be expressed as:

$$\mathbf{a} \times \mathbf{b} = \left(a_x \hat{\mathbf{x}} + a_y \hat{\mathbf{y}} + a_z \hat{\mathbf{z}} \right) \wedge \left(b_x \hat{\mathbf{x}} + b_y \hat{\mathbf{y}} + b_z \hat{\mathbf{z}} \right)$$
$$= \hat{\mathbf{x}} \left(a_y b_z - a_z b_y \right) + \hat{\mathbf{y}} \left(a_z b_x - a_x b_z \right) + \hat{\mathbf{z}} \left(a_x b_y - a_y b_x \right) \qquad (0.26)$$

Note that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$. The cross product is not commutative. (Try it with your fingers.)

The vector cross-product is used in electromagnetism (for example, the force experienced by a moving charge in a magnetic field is given by $\mathbf{F} = q\mathbf{B} \times \mathbf{v}$, where $\mathbf{B}$ is the magnetic field strength, and $\mathbf{v}$ is the velocity of the particle with charge q. It's not used in linear algebra at all.

1.4.4   *Vector Multiplication: The Scalar Triple Product*

This isn't a new way to multiply vectors together, it's just an interesting way to combine cross and dot products when you want to multiply three vectors together. The scalar triple product of three vectors **a**, **b** and **c** is written:

$$\mathbf{a}.\left(\mathbf{b} \times \mathbf{c}\right)$$

Since $\mathbf{b} \times \mathbf{c}$ is the area of the parallelogram with sides **b** and **c**, the scalar triple product is the product of this area with the magnitude of vector **a** times the cosine of the angle between **a** and the area. That's just the volume of the parallelepiped which has the three vectors **a**, **b** and **c** as the non-parallel sides.
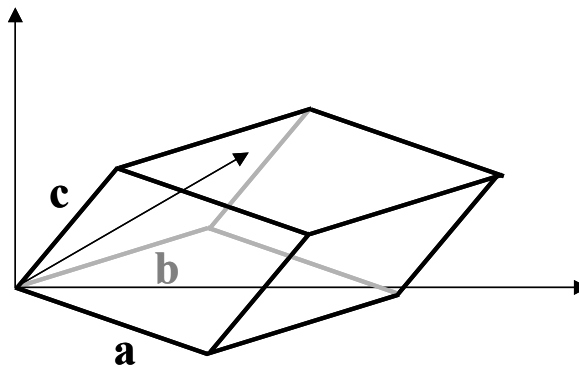
Figure 1-10 The Parallelepiped a,b,c

Clearly, therefore,

$$\mathbf{a}.\left(\mathbf{b} \times \mathbf{c}\right) = \mathbf{b}.\left(\mathbf{c} \times \mathbf{a}\right) = \mathbf{c}.\left(\mathbf{a} \times \mathbf{b}\right)$$

since they are all the same parallelepiped, but since $\mathbf{b} \times \mathbf{c} = -\mathbf{c} \times \mathbf{b}$, we get:

$$\mathbf{a}.\left(\mathbf{b} \times \mathbf{c}\right) = \mathbf{b}.\left(\mathbf{c} \times \mathbf{a}\right) = \mathbf{c}.\left(\mathbf{a} \times \mathbf{b}\right) = -\mathbf{a}.\left(\mathbf{c} \times \mathbf{b}\right) = -\mathbf{b}\left(\mathbf{a} \times \mathbf{c}\right) = -\mathbf{c}.\left(\mathbf{b} \times \mathbf{a}\right)$$

1.4.5   *Vector Division*

You can add and subtract vectors easily, and you can multiply vectors in three different ways (provided the two vectors have the same number of components), but you can't divide them at all. That might seem a bit odd: the cross-product of two vectors is another vector, so if we had three vectors such that:

$$\mathbf{a} \times \mathbf{b} = \mathbf{c}$$

then you might think that it would be possible, if given **b** and **c**, to work out **a** – and surely that's a form of division?

There are two problems with this idea. The first is that the cross-product of two vectors is always perpendicular to the plane containing those vectors. So, if **c** is not perpendicular to **b**, then there will be no value of **a** that satisfies this equation. That's annoying, but it's not in itself a reason why division doesn't work in all cases.

The more serious problem is that there is no unique answer. Given any value of **b** and **c**, any vector **a** that is perpendicular to the vector **c**, and which has a length given by:

$$|\mathbf{a}| = \frac{|\mathbf{c}|}{|\mathbf{b}|\sin\theta}$$

where θ is the angle between **b** and **c** will do. It's not much use having an operation (division in this case) that never gives a unique answer. It's the same problem when trying to divide a scalar by a vector using the dot product – there is no unique answer[11].

1.5 Orthogonal and Orthonormal Vectors

In terms of representing distances in space, *orthogonal vectors* are vectors at right angles to each other. Since the cosine of a right angle is zero, and the dot (inner) product is the product of the amplitudes of the vectors and the cosine of the angle between them:

$$\mathbf{a}.\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta) \tag{0.27}$$

It's very easy to work out whether two vectors are orthogonal or not. Just take the dot product, and if it's zero, then the vectors are orthogonal.

A set of N mutually-orthogonal vectors all of which have N elements, and all of which have a length of one is known as a set of *orthonormal vectors*. This is particularly useful, since any vector with the same number of elements (representing a point in N-dimensional space) can be represented in terms of a weighted sum of the orthonormal vectors.

The simplest example of a set of orthonormal vectors is the co-ordinate system: for example, in three dimensions, the set of vectors:

$$
\begin{aligned}
x_1 &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\
x_2 &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \\
x_3 &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}
\end{aligned}
\tag{0.28}
$$

represent an orthonormal set. Any arbitrary vector with three elements can be expressed in terms of a weighted sum of these vectors, for example:

$$\begin{bmatrix} a & b & c \end{bmatrix} = ax_1 + bx_2 + cx_3 \tag{0.29}$$

Only rarely do we have to consider any other set of orthonormal vectors.

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[11] The third, rarest form of vector multiplication (the reverse operation to the outer product) does have a unique answer when the outer product matrix and one of the component vectors is given, but there aren't many practical uses for this.

1.6 Vectors with Complex Elements

Vectors with real elements can be viewed as a point in N-dimensional space, where N is the number of elements in the vector. That's easy to visualise when N is 1, 2 or 3, and the elements are real, but what about vectors with complex elements? How can we think about those?

One way is to think of the real parts of the elements as one set of N-dimensions, and the complex parts of the elements as forming another set of N-dimensions. So, for example, a vector with three complex elements represents two points in two different three-dimensional spaces: one for the real components, and one for the imaginary components:

$$\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} + j \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \tag{0.30}$$

This makes addition, subtraction, and the inner product 'work' for complex vectors. For example, consider the vectors [1+j 2–j] and [2+j 1–2j]. The dot product is:

$$\begin{bmatrix} 1+j & 2-j \end{bmatrix} \begin{bmatrix} 2+j \\ 1-2j \end{bmatrix} = (1+j)(2+j) + (2-j)(1-2j) = 1-2j \tag{0.31}$$

or in terms of expressing each complex vector as the sum of the two vectors, one real and one imaginary:

$$\left(\begin{bmatrix} 1 & 2 \end{bmatrix} + j \begin{bmatrix} 1 & -1 \end{bmatrix} \right) . \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + j \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right) = \left(\begin{bmatrix} 1 & 2 \end{bmatrix} . \begin{bmatrix} 2 \\ 1 \end{bmatrix} + j^2 \begin{bmatrix} 1 & -1 \end{bmatrix} . \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right)$$
$$+ j \left(\begin{bmatrix} 1 & 2 \end{bmatrix} . \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \end{bmatrix} . \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \tag{0.32}$$
$$= \left(4 + j^2 3 \right) + j(-3+1)$$
$$= 1 - 2j$$

We can define a length for vectors with complex amplitudes, by extending the previous result:

$$\|\mathbf{a}\|^2 = a_x{}^2 + a_y{}^2 + a_z{}^2 \tag{0.33}$$

to use the square of the modulus of the elements, rather than their squares:

$$\|\mathbf{a}\|^2 = \left| a_x \right|^2 + \left| a_y \right|^2 + \left| a_z \right|^2 \tag{0.34}$$

Since for a complex number $x = u + jv$:

$$\left| x \right|^2 = \left| u \right|^2 + \left| v \right|^2 \tag{0.35}$$

The square of this 'amplitude' is the sum of the squares of the length of the vector of real components, and the vector of imaginary components. You could think of this as a sort of $2N$ dimensional space, where the imaginary components are all at right angles to the real components, so this is the square of the 'total length' of the vector.

Asking what the angle is between two vectors with complex components doesn't make a lot of sense, since trying to calculate it using the dot product usually results in the cosine of this angle being a complex number. However, it is useful to define the dot product, and this can readily be done, as shown above.

The less obvious point is that now we've defined the total length of the vector as the square of the modulus of the elements (so that equation (0.34) gives the square of the length of the vector in 2N-dimensional space), we can no longer write:

$$\|\mathbf{a}\|^2 = \mathbf{a}.\mathbf{a} \tag{0.36}$$

This works for vectors with real elements, but not for vectors with complex elements. We need:

$$\|\mathbf{a}\|^2 = |a_x|^2 + |a_y|^2 + |a_z|^2 \tag{0.37}$$

and noting that:

$$|a_x|^2 = a_x^* a_x \tag{0.38}$$

where a_x^* is the complex conjugate of a_x, implies that we can most easily express the square of the modulus of the vector as:

$$\|\mathbf{a}\|^2 = \mathbf{a}^*.\mathbf{a} \tag{0.39}$$

(Again, I'm being lazy with notation, and if $\mathbf{a}$ is a column vector, then to produce the dot (inner) product, I should really write $\mathbf{a}^{*T}$ (or just $\mathbf{a}^H$ for short, it means the same thing) so that the product is a scalar. You can't multiply two column vectors together, or two row vectors.)

While it doesn't make much sense to talk in terms of the angle between two vectors, we can extend the definition of orthogonal vectors to the complex case. This time, we define two vectors as being orthogonal if:

$$\mathbf{a}^H.\mathbf{b} = 0 \tag{0.40}$$

(this time I've stopped being lazy with the notation: $\mathbf{a}$ and $\mathbf{b}$ are column vectors, $\mathbf{a}^H$ is a row vector whose elements are the complex conjugates of the elements of $\mathbf{a}$).

This way, the dot product of a unit length vector (for example $\mathbf{a} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ j \end{bmatrix}$) with itself gives:

$$\mathbf{a}^H.\mathbf{a} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -j \end{bmatrix}.\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ j \end{bmatrix} = \frac{1}{2}(1 \times 1 - j \times j) = \frac{1}{2}(1+1) = 1 \tag{0.41}$$

whereas if we didn't take the complex conjugate, we'd get:

$$\mathbf{a}^T.\mathbf{a} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & j \end{bmatrix}.\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ j \end{bmatrix} = \frac{1}{2}(1 \times 1 + j \times j) = \frac{1}{2}(1-1) = 0 \tag{0.42}$$

which would suggests that the vector is orthogonal to itself! This is clearly silly, and a good reason to define the dot product of two complex vectors as $\mathbf{a}^H.\mathbf{b}$, rather than $\mathbf{a}^T.\mathbf{b}$.

1.7 Problems

1) Calculate the angle between the line from (0,0,0) to (1,2,1) and the line from (0,0,0) to (2,1,2).
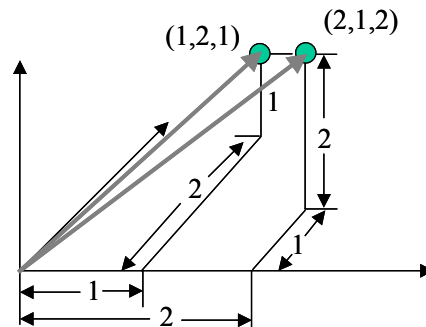
2) Suppose there is an antenna at a point 100 meters above the ground, with a boresight (the direction of maximum power) pointing directly at a point 400 meters north of the base of the antenna mast, the situation is shown in the figure below. By how many degrees is the point X (450 meters north-east of the base of the antenna mast) away from the boresight? (This is an important calculation in working out how much energy is received at this location: antennas are often characterised in terms of the power as a function of angle away from the boresight.)
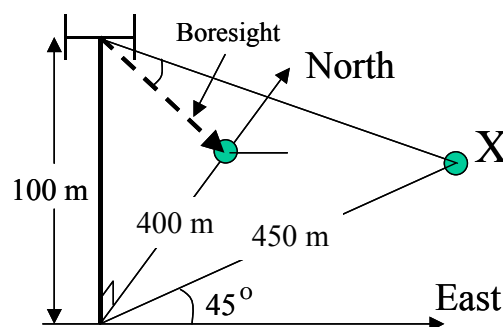
3) A set of orthonormal vectors includes the vectors:

$$\frac{1}{\sqrt{2}}[1 \quad 1 \quad 0] \quad \text{and} \quad \frac{1}{\sqrt{3}}[1 \quad -1 \quad 1]$$

How many vectors are in the set, and what are the other one(s)? How could you express the vectors [1 1 1] and [j 1 1–j] in terms of a linear sum of these orthonormal vectors?

4) Another set of orthonormal vectors includes the vectors:

$$\frac{1}{2}[1+j \quad 1-j \quad 0] \quad \text{and} \quad \frac{1}{2}[j \quad -1 \quad 1+j]$$

Show how you could calculate the other orthonormal vector using the cross-product.