

# 1 WYNTKA... Complex Numbers

“Life is complex... it has a real part, and an imaginary part” – Andrew Koenig.

Complex numbers are an extension of the more familiar world of real numbers that make many practical problems in mathematics and engineering much simpler to solve. This chapter is about these numbers, what they are and why they are useful.

## 1.1 Negative Numbers and Complex Numbers

If you have not come across complex numbers at all before, it might help to start by thinking about negative numbers. They are not obvious either: you can't have a negative number of apples for lunch, or a negative number of coins in your pocket. However, they became very useful in calculating real positive quantities of things, because they can simplify the rules of arithmetic. For example, consider the expression:

$$3 + 4 - 5 + 1 \quad (0.1)$$

It's simple to work out the answer: 3 plus 4 is 7, and 7 minus 5 is 2, and 2 plus 1 is 3, so the answer is 3. There is a rule in arithmetic that states that it doesn't matter in which order you do the additions and subtractions, you should get the same answer. So, for example:

$$3 + 4 + 1 - 5 \quad (0.2)$$

is just 3 plus 4 is 7, and 7 plus 1 is 8, and 8 minus 5 is 3. Same answer. Good.

But what if you try and do:

$$3 - 5 + 4 + 1 \quad (0.3)$$

You start  $3 - 5$  is... and without negative numbers you can't go any further. However, if you have a concept of negative numbers, you can write  $3 - 5$  is  $-2$ , and  $-2$  plus 4 is 2, and 2 plus 1 is 3. Using negative numbers allows us to make the rules of arithmetic more general, and that's a very good thing when we try and program computers to do the maths: we don't need to program them to look at the numbers and put all the positive ones first, for example, which would waste a lot of time (and make the programs more complex, and more likely to contain errors).

It's quite clear where on the number line negative numbers should be. For example, when we're doing  $3 - 5$ , we can just move along the number line in the “negative” direction from three for five steps, and where we end up is minus 2.

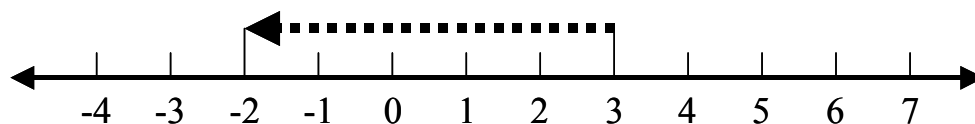


Figure 1-1 Negative Numbers on the Number Line

Complex numbers provide a similar service when working out the answers to certain sums with square roots<sup>1</sup>. If a number is positive, taking the square root is straightforward. However, if a number is negative, what do we do? How can you take the square root of a negative number, when all squares are positive?

## 1.2 Imaginary Numbers

Minus one has no square root: at least no number on the number line, when squared, gives minus one. However, suppose we let the letter  $j$  represent the square root of minus one<sup>2</sup>, so that:

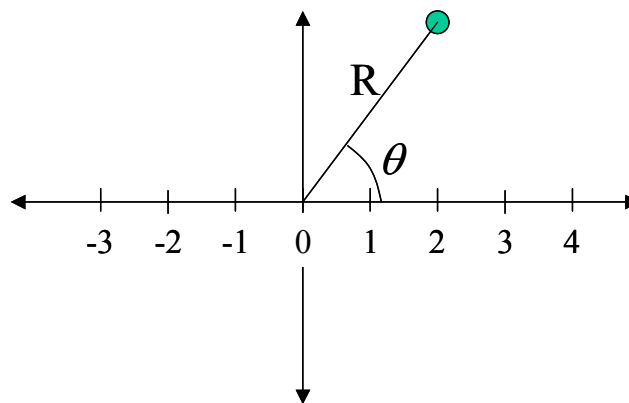
$$j^2 = -1 \quad (0.4)$$

Then the square root of any negative number can be readily expressed in terms of  $j$ , since:

$$\sqrt{-a} = \sqrt{-1}\sqrt{a} = j\sqrt{a} \quad (0.5)$$

As we've noted, there is no number on the number line that when multiplied by itself gives minus one. So  $j$  cannot be on that line: but what if the number line was just one axis of a plane? Could we find another point on that plane that could be used to represent  $j$ ? The answer is yes.

Where would it be? Consider a point in this new number "plane". We could now represent numbers in terms of polar co-ordinates: their distance from the origin  $R$ , and the angle  $\theta$  the line from the origin to the point makes with the positive number axis. In this case, all positive real numbers would have an angle of zero degrees, and all negative real numbers would have an angle of 180 degrees.



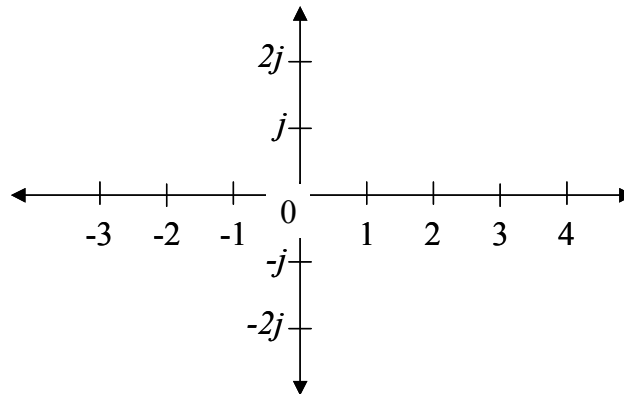
**Figure 1-2 Creating the Argand Diagram**

<sup>1</sup> The first example of this was a formula for working out the roots of cubic equations, published by Gerolamo Cardano in 1545. Even when the roots of the cubic equations were real, it was sometimes necessary to take the square root of a negative number as an intermediate step.

<sup>2</sup> Most mathematicians use  $i$  instead of  $j$  to represent the square root of minus one. It seems to depend on whether you're an engineer (in which case you use  $i$  for currents and apparently think it would be confusing to use the same symbol for both) or not. This is an engineering book, so I'll use  $j$  here.

One could then note that squaring a number results in a new number that has the square of the length of the original number, and twice the angle. (Twice zero is just zero, so all positive numbers remain positive, and twice 180 degrees is 360 degrees, which is all the way around back to the positive real numbers again, so negative numbers have positive squares too.)

This suggests that the line from the origin to  $j$  should have a length of one, and an angle of 90 degrees<sup>3</sup>, so that  $j^2$  would have a length of one, and an angle of 180 degrees: in other words, it would be minus one.



**Figure 1-3** Axes on the Argand Diagram

The multiples of  $j$  are known as the *imaginary numbers*, and on the number plane (known as an *Argand diagram*) they lie on the axis perpendicular to the real number line.

### 1.3 Complex Numbers

As we've seen, one axis on the Argand diagram is the *real axis*, and on this line lie the *real numbers*. The other axis is known as the *imaginary axis*, and on it lie the *imaginary numbers*. Elsewhere in the plane lie the *complex numbers*, and they can be expressed as:

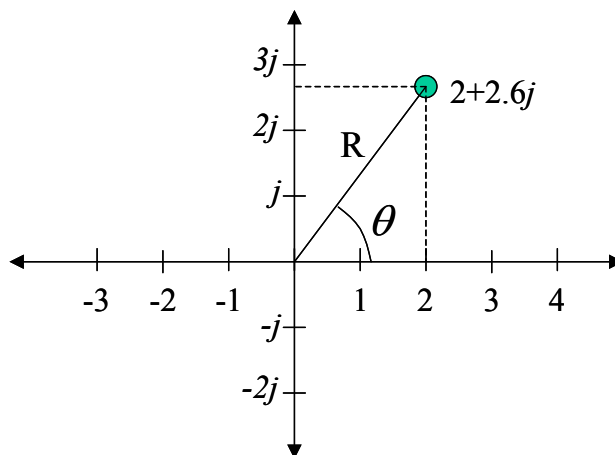
$$z = x + jy \quad (0.6)$$

where  $x$  is the real part of the complex number  $z$ , and  $jy$  is the imaginary part, and  $j$  is the square root of minus one.

Complex numbers are useful for a number of reasons, one being that any polynomial equation that can be written in terms of complex numbers can be solved in the same terms<sup>4</sup>. Another much more useful reason in communications engineering is that they provide a very simple method of doing calculations involving both the amplitude and phase of an oscillation. In this case, we can represent the amplitude of the oscillation as  $R$  (the distance of the complex point from the origin) and the phase of the oscillation as  $\theta$  (the angle between the real axis and the line from the origin to the complex point):

<sup>3</sup> You might notice that using a length of one and an angle of 270 degrees would work just as well. It's just convention that puts  $j$  at 90 degrees.

<sup>4</sup> This is not true of real numbers: for example it is impossible to solve  $x^2 + 3 = 0$  in terms of real numbers.



**Figure 1-4 A Complex Number on the Argand Diagram**

The relationship between  $x$ ,  $y$ ,  $R$  and  $\theta$  is:

$$x + jy = R \exp(j\theta) \quad (0.7)$$

where  $R = \sqrt{x^2 + y^2}$  is known as the *amplitude* or *magnitude* of the complex number, and  $\theta = \tan^{-1}(y/x)$  is known as the *argument* or *phase* of the complex number. (Note that the angle  $\theta$  is considered as going anticlockwise from the real axis.) This somewhat surprising result can be confirmed by considering the Maclaurin series<sup>5</sup> of the complex exponential:

$$\begin{aligned} R \exp(j\theta) &= R \left( 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \dots \right) \\ &= R \left( 1 + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^4}{4!} + \dots \right) + R \left( j\theta + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^5}{5!} + \dots \right) \quad (0.8) \\ &= R \left( 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \right) + jR \left( \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \right) \\ &= R \cos(\theta) + jR \sin(\theta) \end{aligned}$$

and it's clear from the diagram that  $x = R \cos \theta$ , and  $y = R \sin \theta$ . This turns out to be an extremely important and useful result. In the case where  $R = 1$ , it is known as Euler's formula<sup>6</sup>:

$$\exp(j\theta) = \cos(\theta) + j \sin(\theta) \quad (0.9)$$

<sup>5</sup> For those not familiar with the Maclaurin series, see the chapter on Series.

<sup>6</sup> Euler is pronounced "Oil-err". Named after Leonhard Euler, Swiss mathematician and physicist (1707 – 1783). He used  $i$  instead of  $j$  for imaginary numbers, as he wasn't an engineer, and in any case, electrical current hadn't been discovered yet.

Two very useful results can be derived directly from Euler's formula:

$$\begin{aligned}\cos(\theta) &= \frac{\exp(j\theta) + \exp(-j\theta)}{2} \\ \sin(\theta) &= \frac{\exp(j\theta) - \exp(-j\theta)}{2j}\end{aligned}\tag{0.10}$$

### 1.3.1 Sum and Difference Formulas

This relationship between trigonometric function and complex numbers makes it very simple to prove the formulas for sine and cosine of the sum or difference between two angles. Note from Euler's formula above, that:

$$\Re(\exp(j\theta)) = \cos(\theta)\tag{0.11}$$

and:

$$\Im(\exp(j\theta)) = j \sin(\theta)\tag{0.12}$$

where  $\Re(z)$  is the real part of the complex number  $z$ , and  $\Im(z)$  is the imaginary part of  $z$ .

Then, for example:

$$\begin{aligned}j \sin(A + B) &= \Im(\exp(j(A + B))) \\ &= \Im(\exp(jA)\exp(jB)) \\ &= \Im((\cos(A) + j \sin(A))(\cos(B) + j \sin(B))) \\ &= \Im\left(\begin{array}{l} \cos(A)\cos(B) + j \sin(A)\cos(B) \\ + j \sin(B)\cos(A) - \sin(A)\sin(B) \end{array}\right) \\ &= j(\sin(A)\cos(B) + \sin(B)\cos(A))\end{aligned}\tag{0.13}$$

Simple, isn't it? Complex numbers can be really useful. The formulas for  $\sin(A - B)$ ,  $\cos(A + B)$  and  $\cos(A - B)$  are just as easy, but I'll leave those for the examples.

## 1.4 Arithmetic with Complex Numbers

Complex numbers can be added, divided, subtracted and multiplied by considering them to be the sum of a real and an imaginary part. Functions of complex numbers can be integrated and differentiated, however these operations are more complicated to perform.

For example:

$$\begin{aligned}(a + jb) + (x + jy) &= (a + x) + j(b + y) \\ (a + jb) * (x + jy) &= (ax - by) + j(bx + ay)\end{aligned}\tag{0.14}$$

There's a trick to dividing them: you multiply both the numerator (top) and denominator (bottom) by a complex number with the same real part, but minus one times the imaginary part as the denominator. This is the *complex conjugate* of the denominator.

$$\frac{(a + jb)}{(x + jy)} = \frac{(a + jb)(x - jy)}{(x + jy)(x - jy)} = \frac{(ax + by)}{(x^2 + y^2)} + j \frac{(bx - ay)}{(x^2 + y^2)} \quad (0.15)$$

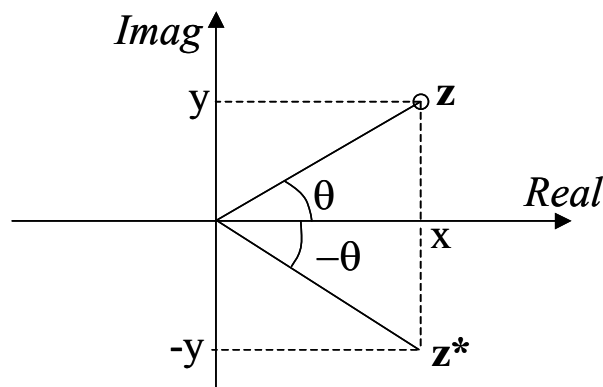
Multiplying and dividing in terms of the magnitude and phase of complex numbers is rather easier:

$$R \exp(j\theta) S \exp(j\phi) = RS \exp(j(\theta + \phi))$$

$$\frac{R \exp(j\theta)}{S \exp(j\phi)} = \frac{R \exp(j(\theta - \phi))}{S} \quad (0.16)$$

### 1.4.1 Complex Conjugates

The complex conjugate of a complex number is a complex number with the same real part, but an imaginary part that is equal in magnitude but opposite in sign. In other words, the complex conjugate of  $(x + jy)$  is  $(x - jy)$ .



**Figure 1-5 Complex Conjugates**

The complex conjugate of  $z$  is written as  $z^*$ . As can be seen from the diagram:

$$z = x + jy = R \exp(j\theta)$$

$$z^* = x - jy = R \exp(-j\theta) \quad (0.17)$$

The complex conjugate has the same magnitude as the original complex number, but the opposite phase.

This leads to the useful result that:

$$z z^* = R \exp(j\theta) R \exp(-j\theta) = R^2 \quad (0.18)$$

The product of a complex number and its conjugate is the square of the amplitude of the complex number. This is a very important and useful result: if the complex number is representing the phase and amplitude of an oscillation, then the square of the amplitude is proportional to the power in the signal.

Any formula using complex numbers remains valid when written using conjugates, and conjugates can be formed by simply reversing the sign of every  $j$  in the formula, for example, the formula for the Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-j\omega t) dt \quad (0.19)$$

could be written in complex conjugate form as:

$$F^*(\omega) = \int_{-\infty}^{\infty} f^*(t) \exp(j\omega t) dt \quad (0.20)$$

## 1.5 Some Useful Results using Complex Numbers

Using complex numbers and the properties of the complex conjugate, there are some useful results we can derive. I'll do these in the form of a list, with a few notes about where they come from. The proofs are straightforward, see the problems and solutions for examples. In all cases,  $z$  and  $w$  are complex numbers.

First, and perhaps most useful, as we've already seen:

$$zz^* = |z|^2 \quad (0.21)$$

or in other words, the product of a complex number with its complex conjugate is the square of the modulus of the complex number. See above (section 1.4.1) for the proof of this.

Secondly, the first in a series of related results:

$$\Re\{wz\} = \Re\{w\}\Re\{z\} + \Im\{w\}\Im\{z\} \quad (0.22)$$

or in words: the real part of the product of two complex numbers is the sum of the product of the real parts and the product of the imaginary parts. (It's very simple to prove: just let  $z = x + jy$  and  $w = u + jv$  and multiply out  $(x + jy)(u + jv)$ .)

You might notice that if  $w$  and  $z$  have unit amplitude, then this result is equivalent to:

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \quad (0.23)$$

Equally straightforward to derive are the similar results (similarly related to the formulas for  $\cos(A - B)$ ,  $\sin(A + B)$  and  $\sin(A - B)$ ):

$$\Re\{wz^*\} = \Re\{w\}\Re\{z\} - \Im\{w\}\Im\{z\} \quad (0.24)$$

$$\Im\{wz\} = \Re\{w\}\Im\{z\} + \Im\{w\}\Re\{z\} \quad (0.25)$$

$$\Im\{wz^*\} = -\Re\{w\}\Im\{z\} + \Im\{w\}\Re\{z\} \quad (0.26)$$

## 1.6 Using Complex Numbers to Represent Oscillations

Just like two-dimensional vectors, complex numbers can be used to represent sinusoidal oscillations: in fact, this is the most common use for complex numbers in communications engineering.

The slightly confusing thing is that there are two common ways of doing this, both of which are frequently used. The first way is to represent the oscillation as the real part of the product of a complex number and a complex exponential oscillation of the form  $\exp(j\omega t)$  where  $\omega$  is the angular velocity (in radians per second) of the oscillation.

Note that:

$$\Re\{\exp(j\omega t)\} = \Re\{\cos(\omega t) + j\sin(\omega t)\} = \cos(\omega t) \quad (0.27)$$

where  $\Re\{z\}$  represents the real part of the complex number  $z$ . This oscillation has an amplitude of one and a phase of zero at time  $t = 0$ .

We can represent any oscillation with this same frequency, perhaps with phase  $\theta$  and amplitude  $R$ , by a complex number  $R \exp(j\theta)$ , by noting that:

$$\begin{aligned} \Re\{R \exp(j\theta) \exp(j\omega t)\} &= \Re\{R \exp(j(\omega t + \theta))\} \\ &= \Re\{R \cos(\omega t + \theta) + jR \sin(\omega t + \theta)\} \\ &= R \cos(\omega t + \theta) \end{aligned} \quad (0.28)$$

This provides a simple way to add together oscillations at the same frequency: you just represent all the oscillations in terms of complex numbers, and then add up the complex numbers. (This is the technique used in the equivalent baseband representation of passband signals, and is frequently used in design and analysis of wireless modulation systems.)

The second useful way to represent oscillations in terms of complex numbers is to represent a real oscillation as the sum of two complex oscillations: this avoids having to take the real part of the expression. For example, an oscillation of amplitude  $R$  and phase  $\theta$  could be represented as:

$$R \cos(\omega t + \theta) = \frac{R \exp(j(\omega t + \theta)) + R \exp(-j(\omega t + \theta))}{2} \quad (0.29)$$

Note that the second term here has a negative frequency: this represents a point moving in a circle around the Argand diagram in a clockwise direction. The first term has a positive frequency, and represents a point in an anti-clockwise direction. The imaginary parts of the two oscillations always cancel out, just leaving the real part:

$$R \exp(j(\omega t + \theta)) = R \cos(\omega t + \theta) + jR \sin(\omega t + \theta) \quad (0.30)$$

$$R \exp(-j(\omega t + \theta)) = R \cos(\omega t + \theta) - jR \sin(\omega t + \theta) \quad (0.31)$$

since cosine is an even function and  $\cos(-x) = \cos(x)$ . Therefore:



$$R \exp(j(\omega t + \theta)) + R \exp(-j(\omega t + \theta)) = 2R \cos(\omega t + \theta) \quad (0.32)$$

This is the technique used in Fourier analysis, and signals and systems theory, and signal processing.

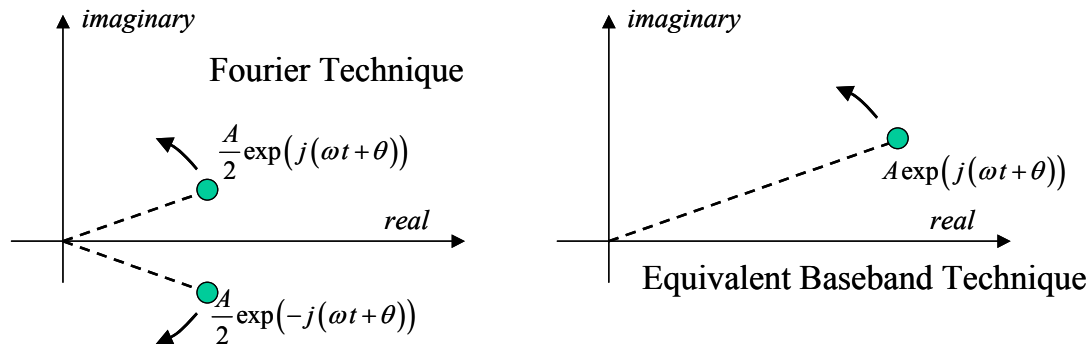


Figure 1-6 Representing Oscillations Using Complex Numbers

## 1.7 Problems

1) If  $z_1 = 2+3j$  and  $z_2 = 4-j$ , what are:

- $z_1 + z_2$
- $z_1 - z_2$
- $z_1 z_2$
- $z_1 / z_2$

2) Evaluate  $1 + \exp(j\pi)$

3) Express  $\sqrt{2} \exp\left(-j\frac{5\pi}{4}\right)$  in terms of its real and imaginary components.

4) Starting from Euler's formula, prove that:

$$\cos(\theta) = \frac{\exp(j\theta) + \exp(-j\theta)}{2}$$

$$\sin(\theta) = \frac{\exp(j\theta) - \exp(-j\theta)}{2j}$$

5) Starting from Euler's formula, prove that:

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

6) Prove that  $\Re\{wz\} = \Re\{w\}\Re\{z\} + \Im\{w\}\Im\{z\}$ .

7) Which trigonometric identity can be easily derived from the following equation, and how?

$$\Im\{wz^*\} = -\Re\{w\}\Im\{z\} + \Im\{w\}\Re\{z\}$$

8) What would you get if you add a cosine wave of amplitude  $\sqrt{10}$ , frequency 10 Hz and phase 18.4 degrees to a cosine wave of amplitude  $\sqrt{2}$ , frequency 10 Hz and phase 315 degrees? (Hint – express both in terms of a complex number, and then add the real and imaginary parts of the two complex numbers.)