1 WYNTKA… Series

Infinite and finite series have many applications in engineering; most commonly they are used to develop simple formulas for complex situations, either by allowing the sum of a large number of expressions to be written in a simple closed form, or by providing a technique to produce approximate formulas that are faster and easier to use.

For example: suppose you're waiting for a chance to transmit a frame. The probability that you can transmit the frame in any one short time interval is a constant, *p*. What's the mean length of time you have to wait before you can start transmitting?

Using the formula for the mean[1]:

$$\bar{x} = \sum_{x=0}^{\infty} p(x)x \tag{0.1}$$

we can easily derive that:

$$\bar{x} = \sum_{x=0}^{\infty} p(1-p)^x x \tag{0.2}$$

since the probability of having to wait for *x* slots is the probability that you haven't been able to transmit the frame for the previous *x* slots $(1 - p)^x$, times the probability that you can transmit the probability in the next slot (*p*).

This formula is much easier to use when expressed in a closed form (i.e. without the summation sign). The study of series yields the useful result:

$$\sum_{x=0}^{\infty} p(1-p)^x x = \frac{1-p}{p} \tag{0.3}$$

which allows the mean to be written as:

$$\bar{x} = \frac{1-p}{p} \tag{0.4}$$

Much better.

Another example: suppose you have derived the formula

$$y = \left(1 - \cos\left(\sqrt{\theta}\right)\right) \tag{0.5}$$

If you're trying to work that out using only a small microcontroller, it'll take a very large number of clock cycles: it's difficult to work out square roots and cosines in software when the only things your hardware can do is addition, subtraction and multiplication.

[1] See the chapter on statistics if you're not sure where this formula comes from.

However, if you know that θ is always a small angle, then you could use a useful series expansion of the cosine function:

$$\cos(\beta) = 1 - \frac{\beta^2}{2} + \frac{\beta^4}{4} - \frac{\beta^6}{6} + \ldots \tag{0.6}$$

and therefore:

$$\cos\left(\sqrt{\theta}\right) = 1 - \frac{\theta}{2} + \frac{\theta^2}{4} - \frac{\theta^3}{6} + \ldots \tag{0.7}$$

and if θ is sufficiently small that terms in θ^2 and higher powers can be neglected, then we could approximate formula (0.5) as:

$$y \approx \frac{\theta}{2} \tag{0.8}$$

and that's much easier to calculate. A plot of equations (0.5) and (0.8) is given below in Figure 1-1. As you see, the much simpler result is a very good approximation over a wide range of angles.

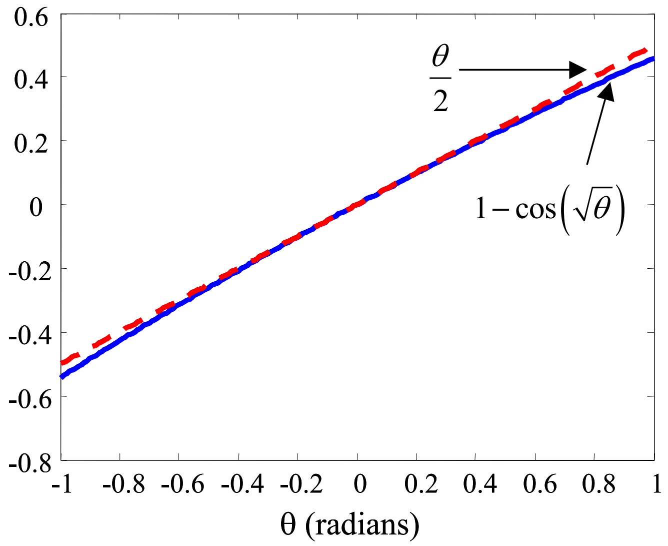

Figure 1-1 Comparison of Exact and Approximate Series Representations

In any application where the angle θ is small, this simpler formula will probably be accurate enough, and using this formula (derived from a series representation of the cosine function) can save a lot of processing time.

1.1 Series, Sequences and Progressions

First, a bit of terminology: a *sequence* or *progression* (they both mean the same thing) is a list of numbers in a well-defined order. Usually, each term in a sequence can be calculated in a simple way from the previous terms, or from its position in the sequence, or both.

As a simple example, consider the sequence:

$$3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \ldots$$

The first term is three, and each subsequent term is one more than the previous term. A more interesting example is:

$$1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad \ldots$$

where the first two terms are one, and each subsequent term is the sum of the previous two terms[2].

Sequences can either have an infinite number of terms (they are then known as *infinite sequences*) or a finite number of terms (in which case they are known as *finite sequences*).

A *series* is the sum of the terms in a sequence.

1.2 Types of Sequence

In the simplest useful type of sequence, each term is just the previous term plus a constant. Let the first term be a, and the constant offset be d, then the sequence can be written:

$$a \quad a+d \quad a+2d \quad a+3d \quad a+4d \quad \ldots$$

for example, if $a = 3$ and $d = 2$, we'd get:

$$3 \quad 5 \quad 7 \quad 9 \quad 11 \quad \ldots$$

This is known as an *arithmetic sequence,* or sometimes as an *arithmetic progression* (*AP*).

Next, consider a sequence in which each term is the previous term multiplied by a constant factor r:

$$a \quad ar \quad ar^2 \quad ar^3 \quad ar^4 \quad \ldots$$

for example, if $a = 3$ and $r = 2$, we'd get:

$$3 \quad 6 \quad 12 \quad 24 \quad 48 \quad \ldots$$

This is known as a *geometric sequence*, or sometimes as a *geometric progression* (*GP*).

Now for a more complex example: a sequence in which the n^{th} term is given by the formula nb^n:

$$0 \quad b \quad 2b^2 \quad 3b^3 \quad 4b^4 \quad \ldots$$

[2] The numbers in this sequence are known as the Fibonacci numbers, and this sequence has a lot of interesting mathematical properties – although not many find use in engineering.

1.3 Summing Arithmetic and Geometric Sequences

One of the most useful things you can do with a sequence is work out the sum of all the terms in the sequence – this is known as a series. Finite sequences always have finite sums (providing all the elements of the sequence are finite – which is usually the case). Infinite sequences may or may not have finite sums: if they do, the series is said be *convergent*, if not, the series is said to be *divergent*, and it's not of much use in engineering other to indicate that the system you are investigating probably won't work.

1.3.1 The Sum of Terms in an Arithmetic Sequence

In the general form, the sum of the first *N* terms of an arithmetic sequence can be written as:

$$S = \sum_{n=0}^{N-1} a + nd = a + (a+d) + (a+2d) + (a+3d) + \dots + (a+(N-1)d) \qquad (0.9)$$

and this can be simplified by writing the summation forwards and then backwards, and then adding up all the corresponding pairs of terms:

$$\begin{array}{ccccccccccc}
S & = & a & + & (a+d) & + & (a+2d) & + & \dots & + & (a+(N-1)d) \\
S & = & (a+(N-1)d) & + & (a+(N-2)d) & + & (a+(N-3)d) & + & \dots & + & a \\
2S & = & (2a+(N-1)d) & + & (2a+(N-1)d) & + & (2a+(N-1)d) & + & (2a+(N-1)d) & + & (2a+(N-1)d)
\end{array}$$

Therefore,

$$\begin{aligned} 2S &= N(2a+(N-1)d) \\ S &= N\left(\frac{a+a+(N-1)d}{2}\right) \end{aligned} \qquad (0.10)$$

In other words, the sum of *N* terms of an arithmetic sequence is *N* times the average value of the first and last terms.

1.3.2 The Sum of Terms in a Geometric Sequence

The geometric sequence has the general form:

$$S = \sum_{n=1}^{N-1} ar^n = a + ar + ar^2 + ar^3 + \dots + ar^{N-1} \qquad (0.11)$$

and this sum can be evaluated by multiplying each term by r, and then subtracting from the initial series:

$$\begin{array}{ccccccccccccc}
S & = & a & + & ar & + & ar^2 & + & \dots & + & ar^{N-1} & & \\
rS & = & & & ar & + & ar^2 & + & \dots & + & ar^{N-1} & + & ar^N \\
(1-r)S & = & a & + & & & & & & & & - & ar^N
\end{array} \qquad (0.12)$$

Therefore:

$$S = \frac{a\left(1-r^N\right)}{(1-r)} \tag{0.13}$$

In the case where r < 1, the geometric series has a limiting value of the sum when an infinite number of terms are considered:

$$S = \frac{a}{(1-r)} \tag{0.14}$$

This is a very useful result – it is worth knowing.

1.4 The Taylor and Maclaurin Series

Many times in engineering we are interested not in the exact answer to a problem, but to an answer that is close enough, and easy to express and work with. The Taylor series is a very useful method to find approximate expressions for more complex functions, an approximation that is especially accurate around one given value of the function.

The idea is to find a polynomial of the form:

$$y(x) = A + B(x-a) + C(x-a)^2 + D(x-a)^3 + E(x-a)^4 + ... \tag{0.15}$$

that has the same value as the target function at the given point *a*, as well as the same gradient, and the same gradient of the gradient (the second differential), and the same gradient of the gradient of the gradient (the third differential), etc. *A*, *B*, *C*, *D*, *E*, and so on are constants.

For example, assume we want to find an approximate expression for $y(x)$ which is accurate around the point $x = a$. We need to find values of *A*, *B*, *C*, etc to use so that the value, gradient, second differential and so on, of our approximate function $y'(x)$ is the same as that of $y(x)$. First, the value of the function at $x = a$ has to be the same as that of $y(x)$, so:

$$\begin{aligned} y(a) = y'(a) &= A + B(a-a) + C(a-a)^2 + D(a-a)^3 + E(a-a)^4 + ... \\ &= A \end{aligned} \tag{0.16}$$

That's the value of *A* determined – it's just equal to the value of the function $y(x)$ at $x = a$. Next, we want to ensure that the gradient of the curve of $y'(x)$ is the same as that of $y(x)$ at this point, and so:

$$\begin{aligned} \left.\frac{dy(x)}{dx}\right|_{x=a} &= \left.\frac{dy'(x)}{dx}\right|_{x=a} = B + 2C(a-a) + 3D(a-a)^2 + 4E(a-a)^3 + ... \\ &= B \end{aligned} \tag{0.17}$$

Continuing to match the second differential, we note that:

$$\begin{aligned} \left.\frac{d^2y(x)}{dx^2}\right|_a &= \left.\frac{d^2y'(x)}{dx^2}\right|_a = 2C + 6D(a-a) + 12E(a-a)^2 + ... \\ &= 2C \end{aligned} \tag{0.18}$$

and therefore:

$$C = \frac{1}{2} \left. \frac{d^2 y(x)}{dx^2} \right|_a \tag{0.19}$$

Carrying on like this, we get the general formula, known as the Taylor series:

$$\begin{aligned} y(x-a) &\approx y(a) + \left. \frac{dy(x)}{dx} \right|_a (x-a) + \frac{1}{2} \left. \frac{dy^2(x)}{dx^2} \right|_a (x-a)^2 + \frac{1}{6} \left. \frac{dx^3(x)}{dx^3} \right|_a (x-a)^3 + \ldots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n y(x)}{d^n x} \right|_a (x-a)^n \end{aligned} \tag{0.20}$$

For the special case where $a = 0$, in other words we want the approximation to be good for very small values of x, the series is known as a Maclaurin series, and the form becomes:

$$\begin{aligned} y(x) &\approx y(0) + \left. \frac{dy(x)}{dx} \right|_0 x + \frac{1}{2} \left. \frac{dy^2(x)}{dx^2} \right|_0 x^2 + \frac{1}{6} \left. \frac{dy^3(x)}{dx^3} \right|_0 x^3 + \frac{1}{24} \left. \frac{dy^4(x)}{dx^4} \right|_0 x^4 + \ldots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n y(x)}{d^n x} \right|_0 x^n \end{aligned} \tag{0.21}$$

1.4.1 Common Maclaurin Series

It's rare to have to work out the terms of a Taylor or Maclaurin series, however it is useful to know at least the first two terms of the most common ones, including:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \tag{0.22}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \tag{0.23}$$

$$\left(1+x\right)^n = 1 + nx + \frac{n(n-1)}{2} x^2 + \frac{n(n-1)(n-2)}{6} x^3 + \ldots \tag{0.24}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \tag{0.25}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \tag{0.26}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \tag{0.27}$$

This is where the well-known small-angle approximations for the trigonometric functions come from: $\sin\theta \approx \theta$; $\tan\theta \approx \theta$; $\cos\theta \approx 1 - \theta^2/2$. (Although please note that these formula apply to angles expressed in radians only: they don't work for angles expressed in degrees.)

Hence, for sufficiently small values of x we can approximate:

$$\sin x \approx x \tag{0.28}$$

$$\cos x \approx 1 - \frac{x^2}{2!} \tag{0.29}$$

$$\left(1+x\right)^n \approx 1 + nx \tag{0.30}$$

$$\frac{1}{1-x} \approx 1 + x \tag{0.31}$$

$$\ln(1+x) \approx x \tag{0.32}$$

$$e^x \approx 1 + x \tag{0.33}$$

What counts as a sufficiently small value of x? That depends on how accurate you need your answer to be. If in doubt, calculate the next term of the series, and see if that changes the answer significantly: if not, then it's a good approximate to make.

1.4.2 The Convergence of Maclaurin Series

We saw before that geometric series can be either convergent or divergent, it depends on the value of the ratio between terms, r. A similar thing is true for most of the Maclaurin series as well: whether they converge or not depends on the value of x.

For certain ranges of x, the series do converge, and when they do, they will converge (eventually) to the correct value[3]. The range of values of x that converge is known as the *convergence radius* of the series.

For example, consider the series for $1 / (1 - x)$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \tag{0.34}$$

Clearly, if $x > 1$, or $x < -1$, this series is not going to converge, but for any value of x with a modulus less than one it does[4]. We could write this as:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \qquad \text{for } -1 < x < 1 \tag{0.35}$$

Whereas the series for $\sin(x)$:

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[3] At least this is usually true for most of the common functions we use in engineering.

[4] You might note that this Maclaurin series can be thought of as a 'geometric series in reverse': it expresses the sum of an infinite number of terms of a geometric progression $1 / (1 - x)$ in terms of the individual terms in the sequence. As expected, it converges for the same range of values of x as the geometric series.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \tag{0.36}$$

converges for any value of x. It's important to be sure the value of x being used is not too big for the series to converge.

For the series given here: $\sin(x)$, $\cos(x)$ and $\exp(x)$ converge for any value of x, but $\ln(1 + x)$, $1 / (1 - x)$, x must be in the range $-1 < x < 1$. For $(1 + x)^n$, any x in the range $-1 < x < 1$ works provided n is positive; if n is negative, then x must be in the range $0 \le x < 1$.

1.5 Problems

1) An arithmetic sequence has 4 terms, first term 2 and last term 8. What is the sum of all the terms in the sequence?

2) A geometric sequence has 4 terms, first term 2 and last term 8. What is the sum of all the terms in this sequence?

3) A geometric sequence has first term one, an infinite number of terms, and each term is half of the previous term. What is the sum of all the terms in this sequence?

4) What is the sum of the sequence $a,\ ar^2,\ ar^4,\ ar^6,\ ar^8,\ \ldots$? For what range of values of r does this sum converge?

5) Consider the series $\sum_{n=0}^{\infty} na^n$. Find a simple expression for the sum of this series. (Hint – proceed in the same way as for the geometric series: multiply each term by a, and then subtract the two series). For what range of values of a does this series have a finite sum?

6) When solving a problem, the formula $y = \ln(1+x)\sin(x)$ is derived. For small values of x, where terms in x^3 and higher powers of x can be neglected, find a simple approximate form of this formula.

7) What is the Maclaurin series of $\cos(x)$ if x is expressed in degrees, rather than radians? Hence produce an approximate expression for $\cos(x)$ with x in degrees for small values of x.

8) Find an approximation for $\ln(x)$ when x is approximately equal to one.

9) If $x = 0.1$, what is the error (in percent) from using the small signal approximations for $\cos(x)$ and $\sin(x)$ given in the notes above, which neglect terms in x^3 and higher powers of x?

10) A right-angled triangle has a hypotenuse of length a, and another side of length b, so that $a > b$. Starting with the Pythagorus theorem, derive a series formula for the length of the third side that does not require any square-roots, using the series:

$$\left(1+x\right)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3 + \ldots$$

where $x = -\left(\frac{a}{b}\right)^2$ and $n = -\frac{1}{2}$.

If b = 21 and $a = 20$, how accurate is your approximate formula? How many terms are required to get within 1% of the correct answer?

What about if b = 21 and $a = \sqrt{41} = 6.4031\ldots$?