

1 What You Need To Know About... Trigonometry

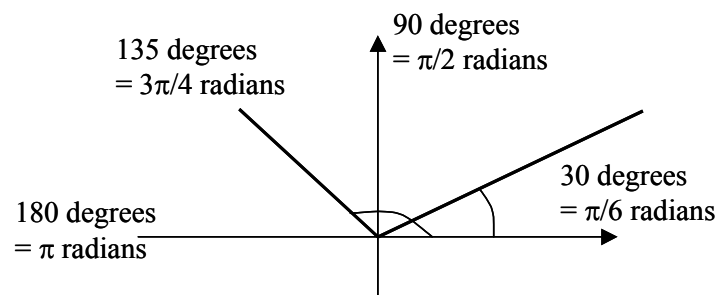
Trigonometry is the arithmetic of angles, in particular angles in *triangles* (hence the name *trigonometry*). The common trigonometric functions, namely sine, cosine and tangent and their inverses, are amongst the most common and most useful function in engineering, and a good understanding and familiarity with them is essential for an engineer.

The trouble with maths is that a lot of the important results depend on other important results which I can't write about yet, because I haven't yet talked about calculus or complex numbers. So – this is just a short introduction to the trigonometric functions. There will be some more details about them in the next few chapters too.

1.1 Degrees and Radians

Before I start, I have to make sure everyone knows the difference between the two common units used to measure angles: degrees and radians.

There are 360 degrees in a circle, so an angle of x degrees is $x/360$ of a circle. There are 2π radians in a circle, so an angle of y radians is $y/2\pi$ of a circle. One radian is $180/\pi$ degrees, or about 57.3 degrees. One degree is $\pi/180$ radians, or about 0.0175 radians.



Degrees are more familiar units to many non-engineers, since it's easy to remember that 90 degrees is a right-angle and 180 degrees is a straight line, and you don't have to worry about how to write any Greek letters. However, radians are more useful in engineering, since:

- With angles measure in radians, the distance moved around a circle is equal to the angle moved times the radius of the circle.
- Conversely, the radius of a curve is equal to the distance moved divided by the change in angle in radians.
- With angles measured in radians, the differential¹ of $\sin(x)$ is just $\cos(x)$, and the differential of $\cos(x)$ is just $-\sin(x)$. This makes calculus much easier.

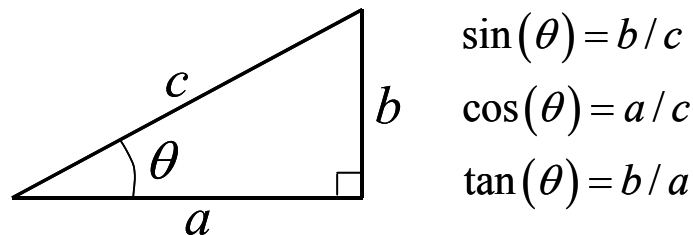
¹ This is one of those times when I've had to skip ahead and refer to something in a later chapter. Sorry. But this really is one of the key advantages of using radians, so I thought I had to mention it.

- Using radians allows the simple approximations for small angles $\sin(x) \approx x$, $\tan(x) \approx x$ and $\cos(x) \approx 1 - x^2/2$.

Unless otherwise specified (and these exceptions will be very rare) when I use an angle, or refer to an angle in a formula, that angle should be expressed in radians.

1.2 Sine, Cosine and Tangent

The functions sine, cosine and tangent, usually written as $\sin(x)$, $\cos(x)$ and $\tan(x)$, and their inverses $\text{asin}(x)$, $\text{acos}(x)$ and $\text{atan}(x)$ are amongst the most common and most useful of all functions². They are all functions of angles, and relate to the ratios of the lengths of the sides of right-angled triangles.



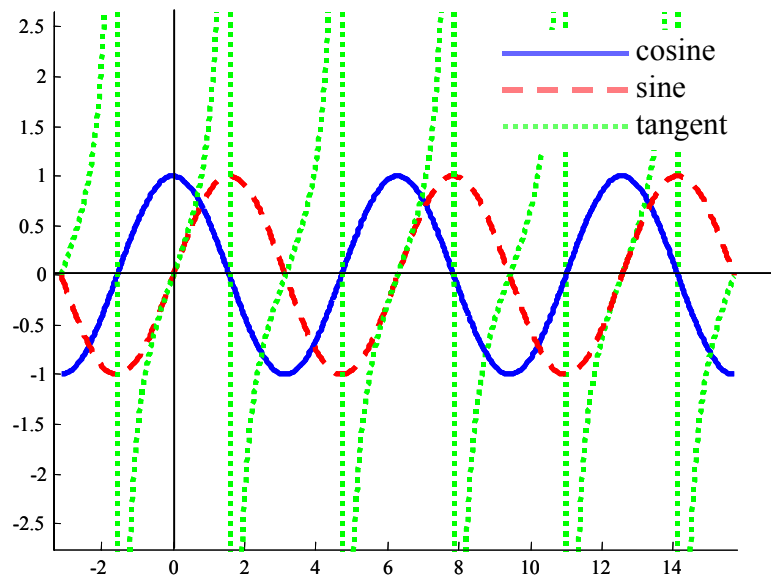
There are many common acronyms to use to remember which is which – the one I learnt was Silly Old Harry Caught A Herring Trawling Off Alaska, which is ridiculous enough to stick in the memory very well. (It stands for SOH-CAH-TOA, or Sine is Opposite / Hypotenuse, Cosine is Adjacent / Hypotenuse, Tangent is Opposite / Adjacent.)

Taking all positive values for length ‘a’ to be to the right of the angle, and positive values of ‘b’ to be to the left of the angle, allows sine and cosine and tangent to be defined for all angles.

These functions, when plotted, look like this:

² Many people write the inverse of the sine function $\sin^{-1}(y)$, so that if $y = \sin(x)$, then $x = \sin^{-1}(y)$. Personally, I find this confusing, since $\sin^2(x)$ is the usual notation for the square of $\sin(x)$, so to be consistent $\sin^{-1}(x)$ should mean $(\sin(x))^{-1} = 1/\sin(x)$.

However as $\sin^{-1}(x)$ is a such a common notation for the inverse sign, when you see it you can invariably assume that it means the inverse sine function.



Note that sine and tangent are odd functions, whereas cosine is an even function. In other words,

$$\sin(-\theta) = -\sin(\theta)$$

$$\cos(-\theta) = \cos(\theta)$$

$$\tan(-\theta) = -\tan(\theta)$$

1.3 Pythagoras' Theorem

One of the most useful of the formulas relating these functions is derived from Pythagoras' theorem. This theorem states that the sum of the squares of the shorter two sides in a right-angled triangle is equal to the square of the longest side (the hypotenuse). In maths:

$$a^2 + b^2 = c^2$$

Therefore³:

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

³ The square of the cosine of θ , $(\cos(\theta))^2$ is usually written $\cos^2(\theta)$ to avoid any possible confusion with $\cos(\theta^2)$, which is the cosine of the square of θ , and an entirely different thing. The same is true for $\sin^2(\theta)$ and $\tan^2(\theta)$ and many other functions.

This is true for any angle⁴. Another formula that can be derived from this, which is less useful, but you might come across occasionally:

$$1 + \tan^2(\theta) = \frac{1}{\cos^2(\theta)} = \sec^2(\theta)$$

Notice the use of the secant function $\sec(\theta)$. $\sec(\theta) = 1 / \cos(\theta)$, just as the cosecant function $\operatorname{cosec}(\theta) = 1/\sin(\theta)$, and the cotangent function $\cot(\theta) = 1 / \tan(\theta)$. I'll try to avoid using them, since most formulas can be expressed just as easily in terms of the more familiar sine, cosine and tangent.

1.4 Sum and Difference Formulas

Very often we need to know the sine of the sum of two angles, or the cosine of the difference between two angles. There are a set of formulas to use to derive these in terms of the sines and cosines of the individual angles:

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$

In particular, note the special cases when $A = B$:

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

$$\sin(2A) = 2\sin(A)\cos(A)$$

The formulas can perhaps be most simply derived from the Euler formula using complex numbers⁵. They are very useful relationships.

Easily derived from these formulas are the formulas for the sum of two cosine waves: again, these are very important and useful results:

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

⁴ There are dozens of different ways to prove this theorem. I came across a webpage that had collected over seventy of them. If it's still there by the time you read this it's at <http://www.cut-the-knot.org>.

⁵ Trigonometry and complex numbers are two subjects that have a large amount of overlap. I'll show how easy it is to prove that these formulas are true in the chapter on complex numbers.

$$\cos(x) - \cos(y) = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

1.5 Oscillations and Phase

It's very common in communications to find a physical quantity that varies in time following the shape of the cosine or sine functions. In this case, we can write these quantities in the form:

$$E(t) = E_0 \cos(\omega t + \theta)$$

The maximum possible value of $E(t)$ is E_0 , and the minimum possible value is $-E_0$. E_0 is known as the *amplitude* of this oscillating quantity. Notice that in this case, the value of E at time $t = 0$ is $E_0 \cos(\theta)$. θ is known as the *phase* of this oscillation.

Cosine is a periodic waveform, it repeats every 2π . Therefore,

$$E_0 \cos(\omega t + \theta + 2\pi) = E_0 \cos\left(\omega\left(t + \frac{2\pi}{\omega}\right) + \theta\right)$$

The value of $E(t)$ at $t + 2\pi/\omega$ is exactly the same as the value at t . $2\pi/\omega$ is known as the *period* of the oscillation, and the inverse of the period, $\omega / 2\pi$, is known as the *frequency*, and is measured in Hertz (Hz). ω itself is known as the *angular frequency*, and is exactly 2π greater than the frequency. The angular frequency is the rate of change of the argument to the cosine function measured in radians per second.

Note that using the results above, we can express this as:

$$\begin{aligned} E &= E_0 \cos(\omega t + \theta) = E_0 \cos(\theta) \cos(\omega t) - E_0 \sin(\theta) \sin(\omega t) \\ &= E_0 \cos(\theta) \cos(\omega t) - E_0 \sin(\theta) \cos\left(\omega t - \frac{\pi}{2}\right) \end{aligned}$$

so that an oscillating quantity with any phase θ can be expressed as the sum of a cosine oscillation with phase zero and amplitude $E_0 \cos(\theta)$, and a cosine oscillation with phase $-\pi/2$ radians (90 degrees) and amplitude $E_0 \sin(\theta)$. This is often a useful thing to do.

Using these trigonometric formulae also allow us to add up two oscillations with the same frequency. For example:

$$E_1 = E_{01} \cos(\omega t + \theta)$$

$$E_2 = E_{02} \cos(\omega t + \phi)$$

$$\begin{aligned} E_1 + E_2 &= E_{01} \cos(\omega t + \theta) + E_{02} \cos(\omega t + \phi) \\ &= E_{01} \cos(\omega t) \cos(\theta) - E_{01} \sin(\omega t) \sin(\theta) \\ &\quad + E_{02} \cos(\omega t) \cos(\phi) - E_{02} \sin(\omega t) \sin(\phi) \\ &= (E_{01} \cos(\theta) + E_{02} \cos(\phi)) \cos(\omega t) - (E_{01} \sin(\theta) + E_{02} \sin(\phi)) \sin(\omega t) \\ &= A \cos(\omega t + \chi) \end{aligned}$$

where:

$$A \cos(\chi) = E_{01} \cos(\theta) + E_{02} \cos(\phi)$$

$$A \sin(\chi) = E_{01} \sin(\theta) + E_{02} \sin(\phi)$$

and the amplitude A and phase χ can be determined by first dividing these two expressions:

$$\tan(\chi) = \frac{E_{01} \sin(\theta) + E_{02} \sin(\phi)}{E_{01} \cos(\theta) + E_{02} \cos(\phi)}$$

and then by adding and squaring them:

$$A^2 \cos^2(\chi) = (E_{01} \cos(\theta) + E_{02} \cos(\phi))^2$$

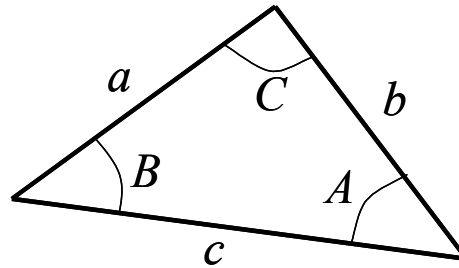
$$A^2 \sin^2(\chi) = (E_{01} \sin(\theta) + E_{02} \sin(\phi))^2$$

$$A = \sqrt{(E_{01} \cos(\theta) + E_{02} \cos(\phi))^2 + (E_{01} \sin(\theta) + E_{02} \sin(\phi))^2}$$

There are easier ways to determine the sum of two oscillations of the same frequency, but this method illustrates directly that adding any two oscillations of the same frequency results in another oscillation of the same frequency – just with a different phase and amplitude.

1.6 Non-Right Angled Triangles

Cosine and sine functions can also be used to work out the angles and lengths of the sides of triangles without a right angle, for example consider the diagram below, showing a triangle with sides of length a, b and c and angles A, B and C .



Perhaps the most useful technique here is the *Cosine Rule*:

$$c^2 = a^2 + b^2 - 2ab \cos(C)$$

which is true in any triangle with three sides a , b and c , with an angle C between the sides with length a and b . However, the *Sine Rule* is also worth knowing:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

which is true in any triangle where A is the angle opposite the side with length a , etc.

There are a lot of different proofs of these formulas also. For this book, all you need to know is how to use them – I'll leave the proofs to the mathematicians.

1.7 Problems

1) Simplify:

- $\cos(\pi - x)$
- $\sin(2\pi - x)$
- $\tan(\pi + x)$
- $(1 - \cos(2x)) / \sin^2(x)$
- $4\sin(x)\cos^2(x) - \sin(3x)$

2) If $\cos(2\pi / x) = 0.5$, find all possible values of x .

3) If $\sin(2x) = \cos(x)$, find all possible values of x .

4) A triangle has sides of length 2, 3 and 4. What is the angle between the two shortest sides?

5) Show that the cosine of 45 degrees must be equal to the sine of 45 degrees by considering the lengths of the sides of a right-angled triangle with two equal sides, and noting that the sum of the angles in any triangle is 180 degrees. Hence, given that the sine of 90 degrees is one, determine the sine of 45 degrees using the formula for $\sin(2x)$.

7) Starting from the formulas for $\cos(A+B)$ and $\cos(A-B)$, prove that:

$$\cos(x) + \cos(y) = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

8) Derive a formula for the difference between two sines in terms of the product of two (or more) cosine or sine functions:

$$\sin(x) - \sin(y) = ?$$